ECE/CS 541: Computer System Analysis

Homework #1 (**Due 09/18/18**)

Student Name: Student's name *Netid: netid*

The intent of this homework is to review material you should have learned in your probability and statistics course (e.g., ECE 313) and explore new material regarding probability theory and stochastic processes we have discussed in class. They vary in difficulty from those you should be able to solve in a couple of minutes, to those that require more thought. If you have trouble with these problems, you should review material from your probability course, or come and talk to us during office hours.

To make things convenient for everyone, you will submit hard copies of your solutions at the start of class on 09/18/18. We encourage you to typeset your solutions using LaTeX though that is not mandatory. Hand written solutions are accepted as long as they are readable.

Exercise 1.

Consider the following program segment :

```
if (c1) {
    do {
        /* some code */
        printf("Hello world!");
    } while (c2);
} else {
    do {
        /* some other code */
        printf("Hello world!");
    } while (c3);
}
```

Assume that $P(c1 == true) = p$, $P(c2 == true) = \frac{3}{5}$, and $P(c3 == true) = \frac{2}{5}$. After running this piece of code over a long period of time, it was estimated that the probability of printing `Hello world!` **exactly three times** is $\frac{3}{25}$. Compute the value of p.

Exercise 2.

A particular webserver may be working or not working. If the webserver is not working, any attempt to access it fails. Even if the webserver is working, an attempt to access it may fail due to network congestion beyond the control of the webserver. Suppose that the a priori probability that the server is working is 0.8. Suppose that if the server is working, then each access attempt is successful with probability 0.9, independently of other access attempts. Find the following quantities :

1. $P\,(\text{first access attempt fails})$
2. $P\,(\text{server is working} \mid \text{first access attempt fails})$
3. $P\,(\text{second access attempt fails} \mid \text{first access attempt fails})$
4. $P\,(\text{server is working} \mid \text{first and second access attempts fails})$

Exercise 3.

Consider the communication network shown in Figure 1. The link capacities in Mbps are given by $C_1 = C_3 = 5$ Mbps, $C_2 = C_5 = 10$ Mbps, and $C_4 = 8$ Mbps, and are the same in each direction. The flow of information from the source s to the destination d can be split into multiple paths. For example,

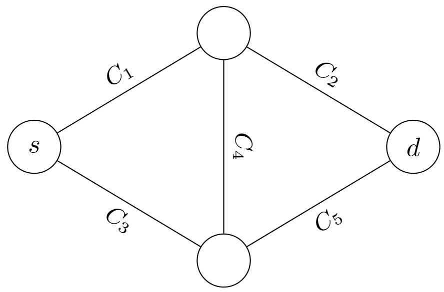

FIGURE 1 – Communication network for Exercise 3.

if all links are working, then the maximum communication rate is 10 Mbps; 5 Mbps can be routed over links 1 and 2, and 5 Mbps can be routed over links 3 and 5.

Let F_i be the event that link i fails. Suppose the F_1, F_2, F_3, F_4, and F_5 are independent and $P(F_i) = 0.2\ \forall i$. Let X be defined as the maximum rate (in Mbps) at which data can be sent from the source node to the destination node. Find the pmf p_X.

Exercise 4.

A mischievous student wants to break into a computer file, which is password protected. Assume that there are n equally likely passwords, and that the student chooses passwords independently and at random, and tries them. Let N_n be the number of trials required to break into the file. Determine the probability density function of N_n if

a) unsuccessful passwords **are not** eliminated from further selections, and
b) unsuccessful passwords **are** eliminated from further selections.

Exercise 5.

Consider a communication channel, where the probability of error-free transmission of a packet has some fixed value p. Suppose further that if a packet is erroneous when received, a retransmission is initiated. This is repeated until an error-free transmission occurs. Assume that the outcome of each transmission is independent of previous transmissions.

Let X be a random variable that represents the number of retransmissions required for error-free reception, i.e., $X = \#$ of retransmissions until the packet is successfully delivered. For example, $X = 0$ if the first packet arrives without errors, $X = 1$ if the first packet fails while the second packet (first retransmission) succeeds.

a) Construct a probability space $(\Omega, \mathcal{F}, P)$ which suffices to serve as the underlying space for X. When defining P, you should be able to give a formula that expresses the probability $P(\{\omega\})$ for any singleton event $\{\omega\} \in \mathcal{F}$. You **do not** have to define P for larger events; their values will follow from the standard axioms of probability.
b) Given an expression for the pmf of X.
c) Formulate the expected value $E[X]$ and the variance $\mathrm{Var}[X]$ of this random variable.
d) Assume we have connected a Linux box to this communication channel and we are running the TCP protocol. We have established a connection to a certain destination and are currently transferring packets to that destination. By default, the Linux machine will try retransmitting the same packet at most 15 times before giving up and closing the connection (If you are on a Linux machine, check out `man tcp` and look for `tcp_retries2`), what is the probability that the machine will close the TCP connection? (It is ok to express the probability as a summation).

Exercise 6.

Let $T_n = (t_1, t_2, \ldots, t_n)$ be a sequence of distinct times for a certain $n > 0$. Let $X = \{X_t \mid t \in T_n\}$ be an independent stochastic process. Define the random variable Y such that

$$Y = \min\{X_{t_1}, X_{t_2}, \ldots, X_{t_n}\}$$

a) Express the cdf of Y in terms of the cdfs of $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$.

b) Assume that the X_{t_i}'s are i.i.d. with an exponential distribution having the parameter $\lambda > 0$. Derive the pdf and the cdf of Y.

Exercise 7.

Let Y be a Poisson random variable with mean $\mu > 0$, and let Z be a geometrically distributed random variable with parameter p such that $0 < p < 1$. Assume that Y and Z are independent.

a) Find $P(Y < Z)$. Express your answer as a simple function of μ and p.

b) Find $P(Y < Z \mid Z = i)$ for $i \geq 1$.

c) Find $P(Y = i \mid Y < Z)$ for $i \geq 0$. Express your answer as a simple function of p, μ, and i.

d) Find $E[Y \mid Y < Z]$, the expected value computed according to the conditional distribution found in part (c). Express your answer as a simple function of p, μ, and i.

Exercise 8.

Let $X \sim N(0,1)$, and define the random variable Y as follows :

$$Y = \begin{cases} X & \text{if } X \in [-10, 10], \\ -X & \text{if } X \notin [-10, 10] \end{cases}$$

Derive the cdf of Y. (*Hint :* Examine the pdf of X. Is there anything you can say about $P(X \leq x)$ and $P(X > -x)$).

Exercise 9.

Let $X = (X_0, X_1, X_2, X_3, \ldots)$ be an independent stochastic process where the X_i's are i.i.d. exponential random variables with parameter $\lambda > 0$. Consider now the *random sum* :

$$S_N = \sum_{i=0}^{N-1} X_i$$

where N is a geometrically distributed random variable with parameter p such that $0 < p < 1$. In other words, for $k \in \{1, 2, 3, \ldots\}$,

$$P(N = k) = p(1-p)^{k-1}$$

Show that S_N is exponentially distributed and compute its parameter.
