What is a random variate?

Outcome of a random variable.

We have seen how to produce a \( U[0,1] \) r.v. but what about more useful ones (as long as they were integers, etc.)

We can apply simple transformations to \( U[0,1] \) to achieve that.

**Example 1**

Let \( U \) be a uniform \([0,1]\) r.v.

**Goal 1:** get \( X \sim \text{Uniform}[a,b] \)

Define \( X = a + (b-a)U \)

Notice \( U = 0 \rightarrow X = a \)
\( U = 1 \rightarrow X = b \)

\[ P(X < x) = P\left\{ a + (b-a)U < x \right\} \]
\[ = P\left\{ U < \frac{x-a}{b-a} \right\} \]
\[ = F_U\left( \frac{x-a}{b-a} \right) \]

**Inverse Transform Method**

Let \( F(x) \), \( x \in \mathbb{R} \) be the CDF

\[ F : \mathbb{R} \rightarrow [0,1] \text{ non-negative, monotonically continuous from the right} \]

\[ F(a) = 1 \quad F(-a) = 0 \]

Objective of random variate generation: generate r.v. \( X \) with distribution \( F(x) \)

i.e., \( P(X \leq x) = F(x) \quad x \in \mathbb{R} \)

The inverse of \( F \), \( F^{-1} : [0,1] \rightarrow \mathbb{R} \)

\[ F^{-1}(y) = \min \{ z : F(z) > y \} \quad y \in [0,1] \]
F^{-1} exists because: continuous f monotone
⇒ for every $x \in \mathbb{R}$ there is a unique $u$ with $F^{-1}(u) = x$

Proposition: Define $X = F^{-1}(U)$ and $U \sim \text{Uniform}[0,1]$ then $X \sim \text{CDF} F$ i.e. $P(X \leq x) = F(x)$

\[
P(X \leq x) = P(F^{-1}(U) \leq x) = P(F[F^{-1}(U)] \leq F(x)) \quad \text{[Due to monotonicity of function of smaller to smaller]}
\]

\[
P(U \leq F(x)) \quad \text{[by def]} \quad f_X(x) = \frac{f_U(F(x))}{f_U(x)} = f(x)
\]

Ex: Generate continuous $Y \sim \text{Uniform}(a,b)$

\[
f_Y(x) = \begin{cases} 
\frac{1}{b-a} & a < x < b \\
0 & \text{otherwise}
\end{cases}
\]

\[
F_Y(x) = \begin{cases} 
0 & x < a \\
\frac{x-a}{b-a} & a \leq x \leq b \\
1 & x > b
\end{cases}
\]

\[
F^{-1}(x) = (b-a)x + a
\]

To generate function $X \sim \text{Uniform}[0,1]$

\[
Y \sim \text{Uniform}[a,b]
\]

\[
Y = (b-a)U + a
\]

Ex: Generate $Y \sim \text{Exponential}(\lambda)$

\[
F_Y(y) = 1 - e^{-\lambda y}
\]

\[
Y = F_Y(y) = 1 - e^{-\lambda y}
\]

\[
1 - y = e^{-\lambda y}
\]

\[
\ln(1-\gamma) = -\frac{\gamma}{\lambda}
\]

\[
y = \frac{-1}{\lambda} \ln(1-\gamma)
\]
Aside

If we have two independent exponentials $X_1 \sim \exp(\lambda_1) \; X_2 \sim \exp(\lambda_2)$ and $\lambda_1 << \lambda_2$.

still possible to get sample from $\exp(\lambda_1) > \exp(\lambda_2)$

But if you want to compare a system with slow exponential rate with one that is faster:

System 1

\[
X_1, i = \frac{1}{\lambda_1} \ln (1 - u,)
\]

System 2

\[
X_{2, i} = \frac{1}{\lambda_2} \ln (1 - u,)
\]

\[
\text{coupled using the same } U_i
\]

\[
X_1, i > X_{2, i}
\]

\[
\text{with Weibull: } \quad F(x) = \begin{cases} 1 - e^{-\left(\frac{x}{\lambda}\right)^k} & x > 0 \\ 0 & x < 0 \end{cases}
\]

\[
\text{such that:}
\]

\[
u = 1 - e^{-\left(\frac{x}{\lambda}\right)^k}
\]

\[
1 - u = e^{-\left(\frac{x}{\lambda}\right)^k}
\]

\[
\ln(1-u) = -\left(\frac{x}{\lambda}\right)^k
\]

\[
- \ln(1-u) = \left(\frac{x}{\lambda}\right)^k
\]

\[
-\lambda^k \ln(1-u) = x^k
\]

\[
x = -\lambda \left[\ln (1-u)\right]^{1/k}
\]
Discrete transformations

\[ F^{-1}(y) = \min \{ x : F(x) \geq y \} \]

\[ U = F(x) \]
\[ F^{-1}(U) = x = \min \{ x : F(x) \geq U \} \]

By round up

**Example:** \( Y \sim \text{Geometric}(p) \)

\[ F_Y(k) = 1 - (1-p)^k+1 \]

\[ F^{-1}(u) = \min \left\{ k : (1-p)^k \leq u \right\} \]

\[ \frac{\log(1-u)}{\log(1-p)} \]

\[ \left\lfloor \frac{\log(1-u)}{\log(1-p)} \right\rfloor + 1 = k \]

- Normal \((0, 1)\) - Standard Normal

\[ F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} \, dz \]

Hard to do because inverting CDF is hard

**Polar coordinates**

\[ \rho^2 = x^2 + y^2 \]
\[ \tan(\theta) = \frac{y}{x} \]
\[ x = \rho \cos \theta \]
\[ y = \rho \sin \theta \]

**Box-Muller Method**

Consider two independent \( N(0,1) \)

\[ X \sim N(0,1) \]
\[ Y \sim N(0,1) \]
What is their joint density function?

\[ f(x, y) = f(x) \cdot f(y) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \cdot \frac{e^{-y^2/2}}{\sqrt{2\pi}} \]

\[ = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2+y^2)} \]

\[ = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2+y^2}{2}} \quad \text{(} y^2 = x^2 + y^2 \text{)} \]

This is still a product of two pdfs but can we think of them as different pdfs from normals, something that we know

\[ \frac{1}{\sqrt{2\pi}} \text{ is pdf of } U(0, 2\pi) \]

What about \( e^{-x^2/2} \)?

Let's look at the CDF of an exponential RV, \( Z \):

\[ P(Z \leq z) = 1 - e^{-z^2/2} \]

\[ \frac{d}{dz} (1 - e^{-z^2/2}) = 2 \cdot \frac{z}{\sqrt{2}} e^{-z^2/2} \]

\[ = e^{-z^2/2} \]

\[ z^2 \sim \exp(\frac{\gamma}{2}) \text{ with pdf } e^{-z^2/2} \]

Remember, we can sample an exponential from uniform by

\[ -\frac{1}{\lambda} \ln(1-u) \sim -\frac{1}{\lambda} \ln(u) \]

So sampling for \( R^2 = -2 \ln(u) \)

\[ R = \sqrt{-2 \ln(u)} \]

Also remember, for Uniform \( (a, b) \), \( (b-a)u + a \sim \text{Uniform}(0, 2\pi) \)

\[ \frac{R^2}{\sqrt{2\pi}} \sim \Theta \]
Now polar \rightarrow rectangular coordinates

\[\begin{align*}
x &= r \cos \theta = (-2 \ln U_1)^{1/2} \cos (U_2 \cdot 2\pi) \\
y &= r \sin \theta = (-2 \ln U_2)^{1/2} \sin (U_2 \cdot 2\pi)
\end{align*}\]

\[x \sim N(\mu, \sigma)\]

Transform \(Z \sim N(0,1) \rightarrow \sigma Z + \mu\)

**Acceptance - Rejection Method**

**Basic form**

a) Sample \(x\) from "easy" distribution
b) accept \(x\) if it meets some criteria | reject otherwise

Simplest form: We want \(X \sim U(\frac{a}{4}, \frac{b}{4})\)

1. Sample \((x, y)\) a point.
   \(x \sim U[a, b]\)
   \(y \sim U[a, d]\)
2. Use geometry to determine whether \((x, y)\) falls within red polygon → accept if \(x, y\)
Generalization

Goal: generate $X$ with density function $f$

Let $t, f$ be density functions
- $t$ is easy to sample
- $f(x)$ can be computed

$t(x)$ must majorize $f(x)$, i.e. $t(x) \geq f(x)$ \quad \forall x$

then $t(x) > 0$ but $\int_{-\infty}^{\infty} t(x) \, dx = \int_{-\infty}^{\infty} f(x) \, dx = 2 \Rightarrow t(x)$ is not a density function

Set $C = \int_{-\infty}^{\infty} t(x) \, dx > 1$ \quad \Rightarrow \text{area under curve}

Define $r(x) = t(x)/C \quad \forall x$

is a density since it integrates to 1

Algorithm

1. generate $y \sim f$ having density $r$
2. generate $u \sim U(0, 1)$
3. $\quad y \leftarrow f(y)/t(y)$ \quad return $y$ and stop
4. else reject $y$ and return to step 2

Note: Since $t$ majorizes $f$

$$f(y)/t(y) \leq 1 \quad \Rightarrow \quad [0, f(y)/t(y)]$$

$$P(\text{accept}) = \int_{0}^{f(y)/t(y)} dx = \frac{1}{C} \cdot 1 = \frac{1}{C}$$

So want $C$ on curve to 1. Best $f/\bar{f}$. 
Convolution Method

- Some random $V_i$ are naturally expressed as sums of others

- Sum of $N$ bernoulli $Y_i$ is binomial -> # of successes $\sum_{i=1}^N X_i$ of $\text{Bern}(p)$

- Sum of $k$ exponentials is Erlang $-k$

Let $Y_i \sim \text{Exp}(\theta)$

\[ Y_i = -\frac{1}{\theta} \ln(U_i) \]

\[ X = \sum_{i=1}^k Y_i \]

\[ X \sim \text{Erlang}(k, \theta) \]

\[ \mathbb{E}[X] = \frac{k}{\theta} \]

We already know using inverse transform how to generate exponentials

\[ Y = \sum_{i=1}^k \frac{1}{\theta} \ln(U_i) \]

\[ Y \sim \text{Erlang}(k, \theta) \]

\[ \mathbb{E}[Y] = \frac{k}{\theta} \ln(n, u) \]