Combinatorial Modeling Methods

Introduction to Combinatorial Methods

- Combinatorial validation methods are the simplest kind of analytical/numerical techniques and can be used for reliability and availability modeling under certain assumptions.

- Assumptions are that component failures are independent, and for availability, repairs are independent.

- When these assumptions hold, simple formulas for reliability and availability exist.

Lecture Outline

- Review definition of reliability
- Failure rate
- System reliability
  - Maximum
  - Minimum
  - k of N
- Reliability formalisms
  - Reliability block diagrams
  - Fault trees
  - Reliability graphs
- Reliability modeling process

Reliability

- One key to building highly available systems is the use of reliable components and systems.
- Reliability: The *reliability* of a system at time t ($R(t)$) is the probability that the system operation is proper throughout the interval $[0,t]$.
- Probability theory and combinatorics can be directly applied to reliability models.
- Let X be a random variable representing the time to failure of a component. The *reliability* of the component at time t is given by

  $R_X(t) = P[X > t] = 1 - P[X \leq t] = 1 - F_X(t)$. Also called the *survivor* function
- Similarly, we can define *unreliability* at time t by

  $U_X(t) = P[X \leq t] = F_X(t)$.

Failure Rate

What is the rate that a component fails at time t? This is the probability that a component that has not yet failed fails in the interval $(t, t + \Delta t)$ divided by Δt, as $\Delta t \to 0$.

Note that we are not looking at $P[X \in (t, t + \Delta t)] = f_X(t)$. Rather, we are seeking $P[X \in (t, t + \Delta t) | X > t]$.

$$P[X \in (t, t+\Delta t) \mid X > t] = \frac{P[X \in (t, t+\Delta t), X > t]}{P[X > t]}$$

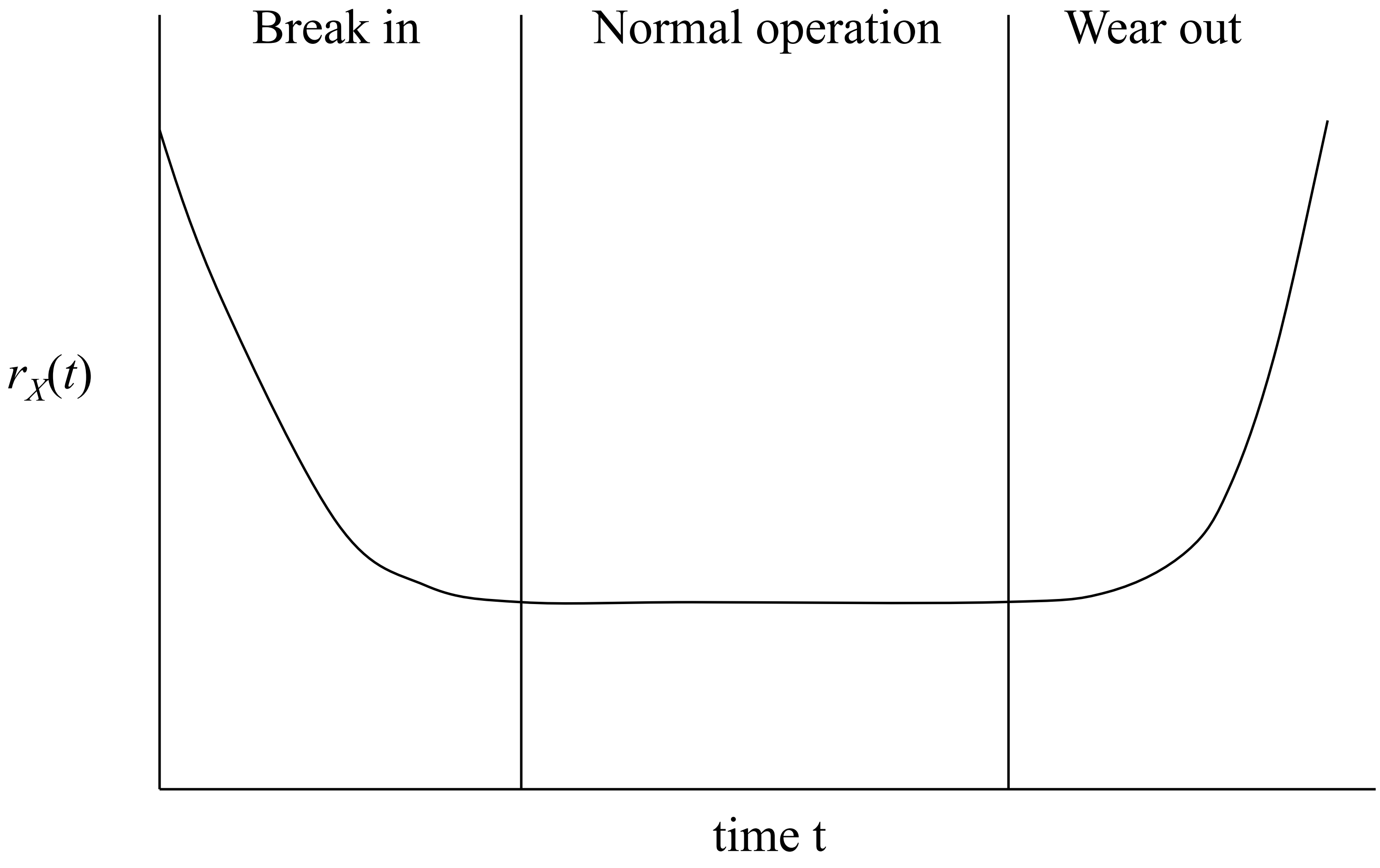
$$= \frac{P\left[X \in \left(t, t+\Delta t\right)\right]}{1 - F_X\left(t\right)}$$

Thinking in terms of rate

$$r_x(t) = \lim_{\Delta t \to 0} \frac{\frac{P[\in (t, t+\Delta t)]}{\Delta t}}{1 - F_X(t)} = \frac{f_X(t)}{\overline{F}_X(t)}$$

$r_X(t)$ is called the *failure rate* or *hazard rate*.

Typical Failure Rate



System Reliability

While F_X can give the reliability of a component, how do you compute the reliability of a system?

System failure can occur when one, all, or some of the components fail. If one makes the *independent failure assumption*, system failure can be computed quite simply. The independent failure assumption states that all component failures of a system are independent, i.e., the failure of one component does not cause another component to be more or less likely to fail.

Given this assumption, one can determine:

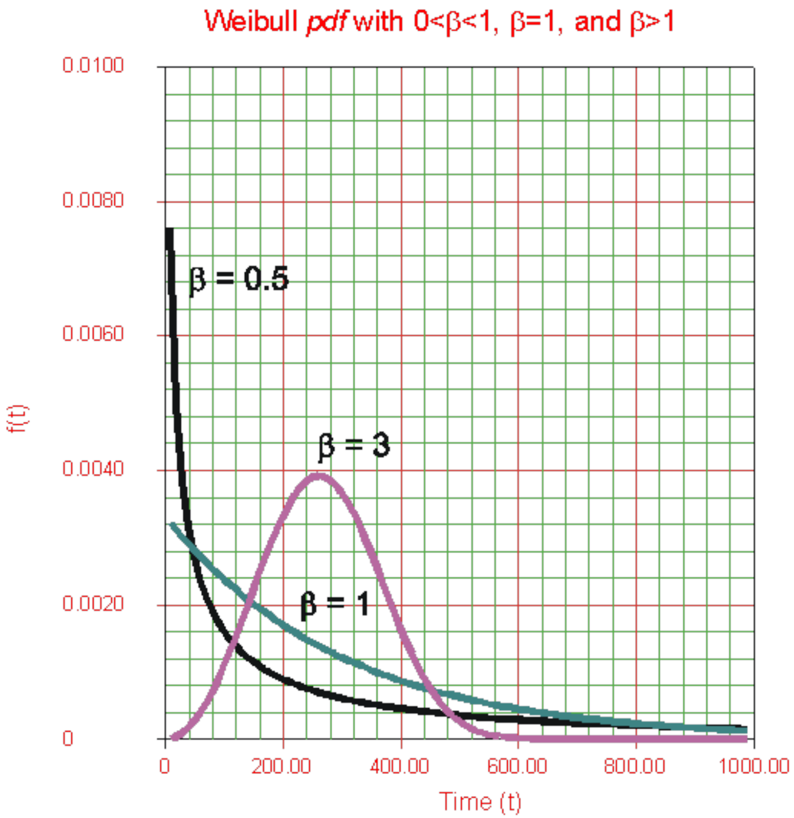
1) Minimum failure time of a set of components
2) Maximum failure time of a set of components
3) Probability that k of N components have failed at a particular time t.

Example: Weibull Distribution

Weibull Distribution

- Often used to model lifetime data
- β called the shape parameter
- η is the scale parameter
- ϒ is the location parameter

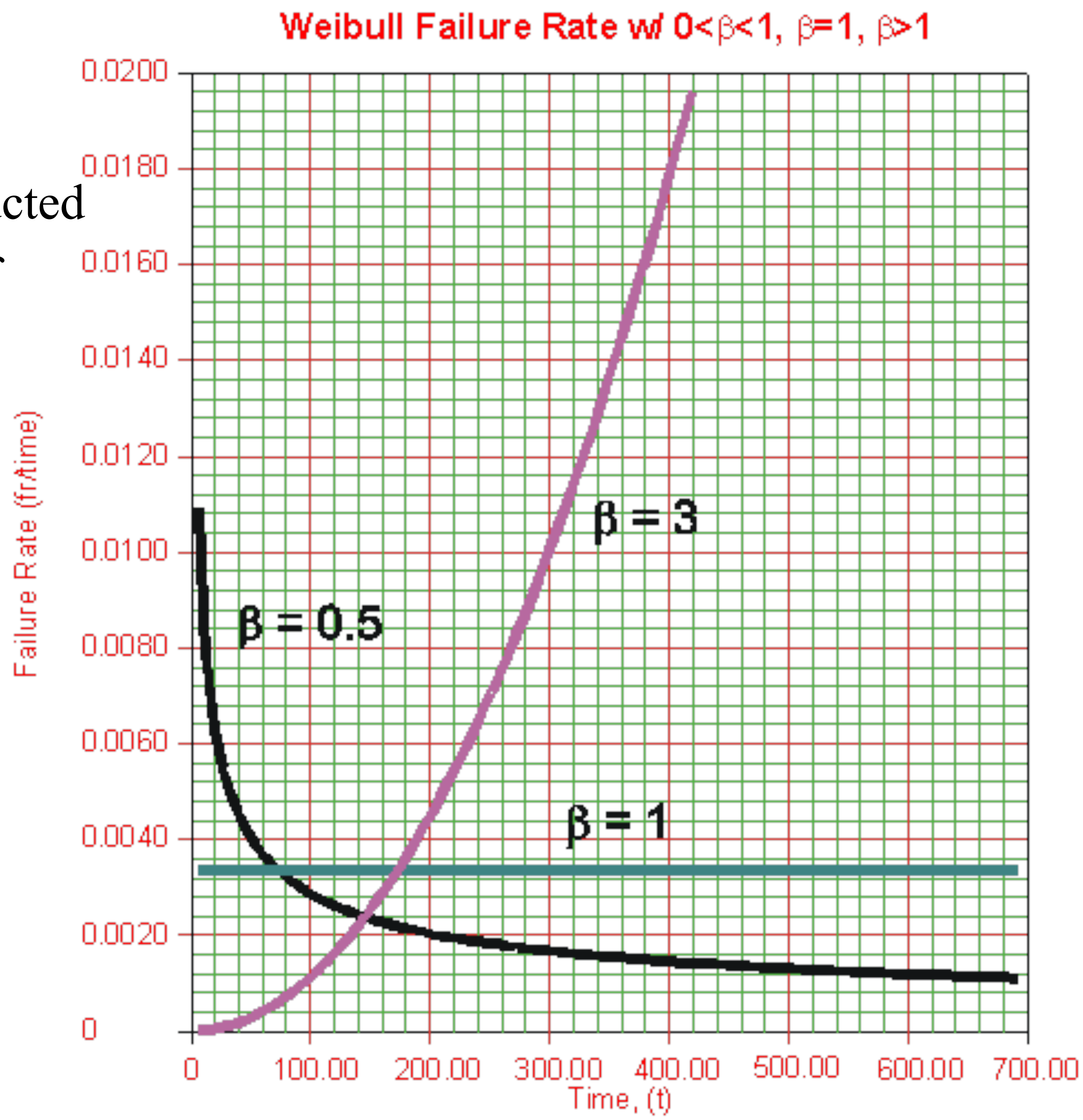
$$f(t) = \frac{\beta}{\eta}\left(\frac{t-\gamma}{\eta}\right)^{\beta-1} e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}}$$



Example: Weibull Distribution

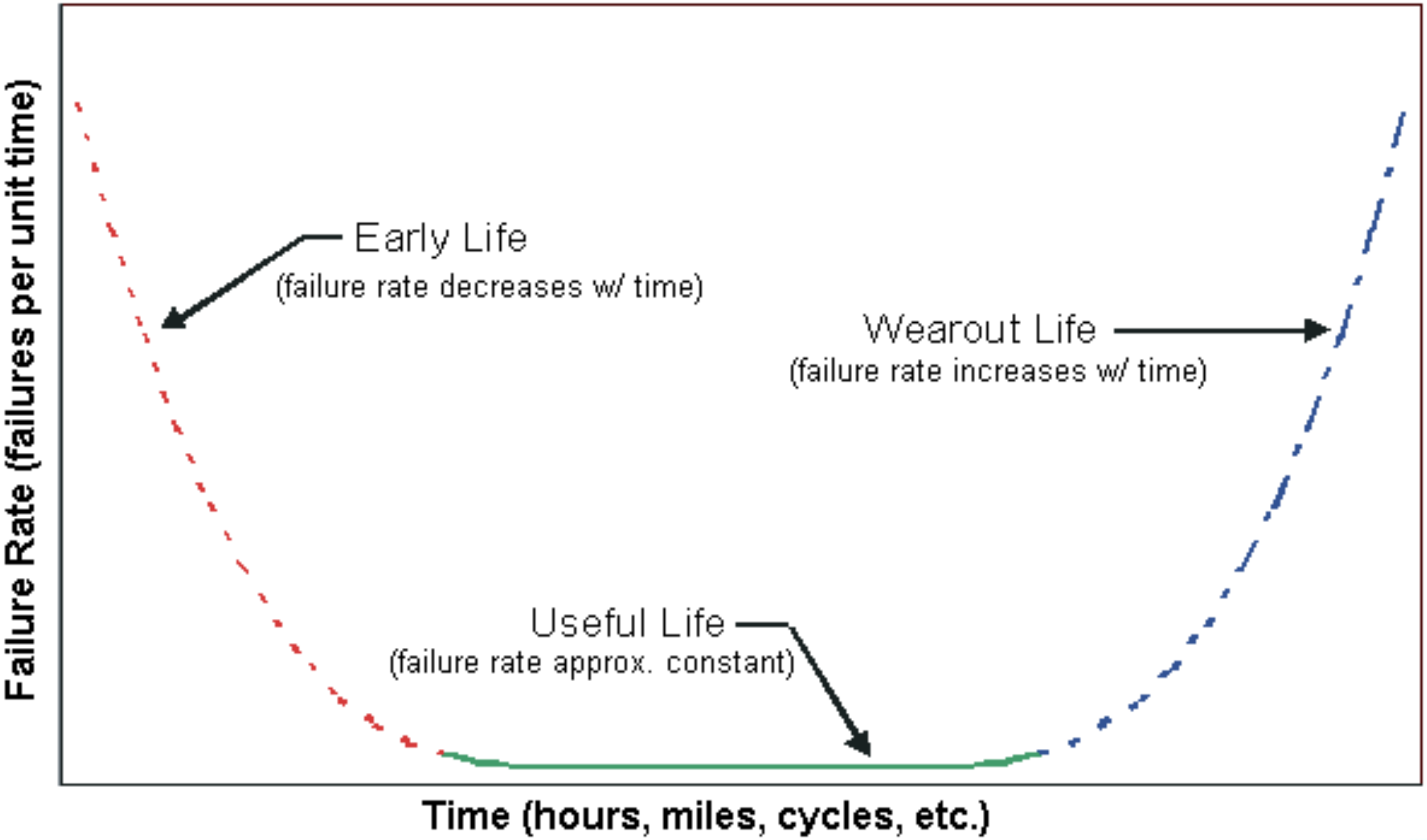
Weibull Distribution

- Impact of β
- Bathtub function constructed by piece-wise definition of Weibulls with different β



Example: Weibull Distribution

Weibull Distribution



Example: Weibull Distribution

Weibull Distribution

$$f(t) = \frac{\beta}{\eta}\left(\frac{t-\gamma}{\eta}\right)^{\beta-1} e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}}$$

$$\overline{F}(t) = e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}}$$

$$r_X(t) = \frac{f_X(t)}{\overline{F}(t)} = \frac{\beta}{\eta}\left(\frac{t-\gamma}{\eta}\right)^{\beta-1}$$

Observe special case of $\gamma = 0, \quad \beta = 1 \qquad r_X(t) = \frac{1}{\eta}$

Maximum of *n* Independent Failure Times

Let $X_1, \ldots, X_n$ be independent component failure times. Suppose the system fails at time S if S is the earliest time at which all components are in the failed state.

Thus, $S = \max\{X_1, \ldots, X_n\}$

What is $F_s(t)$?

$$
\begin{aligned}
F_s(t) &= P[S \le t] \\
&= P[X_1 \le t \text{ AND } X_2 \le t \text{ AND } \ldots \text{ AND } X_n \le t] \\
&= P[X_1 \le t]\, P[X_2 \le t] \ldots P[X_n \le t] && \text{By independence} \\
&= F_{X_1}(t) F_{X_2}(t) \ldots F_{X_n}(t) && \text{By definition} \\
&= \prod_{i=1}^{n} F_{X_i}(t)
\end{aligned}
$$

Minimum of *n* Independent Component Failure Times

Let $X_1, \ldots, X_n$ be independent component failure times. A system fails at time S if S is the earliest time at which any component is in the failed state.

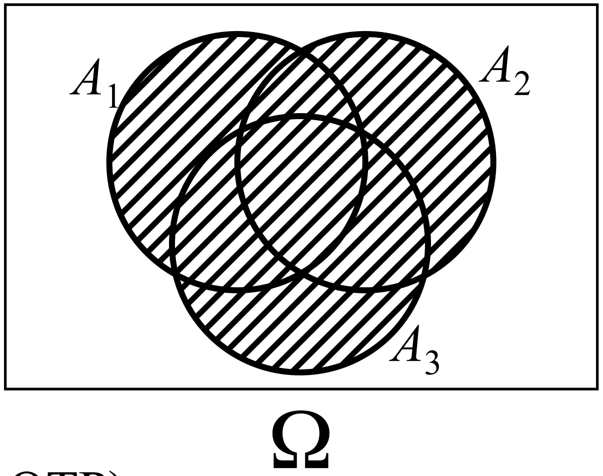
Thus, $S = \min\{X_1, \ldots, X_n\}$.

What is $F_S(t)$?

$$F_S(t) = P[S \le t] = P[X_1 \le t \text{ OR } X_2 \le t \text{ OR } \ldots \text{ OR } X_n \le t]$$

Trick: If A_i is an event, and $\overline{A}_i$ is the set complement such that $A_i \cup \overline{A}_i = \Omega$ and $A_i \cap \overline{A}_i = \varnothing$, then

$$P[A_1 \text{ OR } A_2 \text{ OR} \ldots \text{OR } A_n]$$
$$= 1 - P[\overline{A}_1 \text{ AND } \overline{A}_2 \text{ AND} \ldots \text{AND } \overline{A}_n]$$



This is an application of the law of total probability (LOTP).

Minimum cont.

$$F_s(t) = P[X_1 \le t \text{ OR } X_2 \le t \text{ OR } \ldots \text{ OR } X_n \le t]$$

$$= 1 - P[X_1 > t \text{ AND } X_2 > t \text{ AND } \ldots \text{ AND } X_n > t] \qquad \text{By trick}$$

$$= 1 - P[X_1 > t]\, P[X_2 > t] \ldots P[X_n > t] \qquad \text{By independence}$$

$$= 1 - (1 - P[X_1 \le t])(1 - P[X_2 \le t]) \ldots (1 - P[X_n \le t]) \qquad \text{By LOTP}$$

$$= 1 - \prod_{i=1}^{n} (1 - F_{X_i}(t))$$

k of N

Let $X_1, \ldots, X_n$ be component failure times that have identical distributions (i.e., $F_{X_1}(t) = F_{X_2}(t) = \ldots$). The system fails at time S if S is the earliest time at which any k of the N components are in the failed state.

$F_S(t)$ = P[<u>at least</u> k components failed by time t]

= P[exactly k failed OR exactly $k + 1$ failed OR . . . OR exactly N failed]

= P[exactly k failed] + P[exactly $k + 1$ failed] + . . . + P[exactly N failed]

- by independence and axiom of probability.

What is P[exactly k failed]?

= P[k failed and (N - k) have not]

$$= \binom{N}{k} F_X(t)^k (1 - F_X(t))^{N-k}$$

where $F_X(t)$ is the failure distribution of each component.

Thus, $F_S(t) = \sum_{i=k}^{N} \binom{N}{i} F_X(t)^i (1 - F_X(t))^{N-i}$

k of N in General

For non-identical failure distributions, we must sum over all combinations of at least k failures.

Let G_k be the set of all subsets of $\{X_1, \ldots, X_N\}$ such that each element in G_k is a set of size at least k, i.e.,

$$G_k = \{g_i \subseteq \{X_1, \ldots, X_N\} : |g_i| \geq k\}.$$

The set G_k represents all the possible failure scenarios.

Now F_S is given by

$$F_S(t) = \sum_{g \in G_k} \left(\prod_{X \in g} F_X(t) \right) \left(\prod_{X \notin g} \left(1 - F_X(t)\right) \right)$$

Component Building Blocks

Complex systems can be analyzed hierarchically.

Example: A computer fails if both power supplies fail or both memories fail or the CPU fails.

System problem is one of a minimum : the system fails when the first of three subsystems fails…proper formulation is

– Power supply subsystem is a maximum : both must fail
– Memory subsystem is a maximum : both must fail

$$F_S(t) = 1 - (1 - F_{P1}(t)F_{P2}(t))\,(1 - F_{M1}(t)F_{M2}(t))\,(1 - F_C(t))$$

Probability at least 1 power source is up at t

Probability all 3 subsystems are up at t

Summary

A system comprises N components, where the component failure times are given by the random variables $X_1, \ldots, X_N$. The system fails at time S with distribution F_S if:

Condition:	Distribution:
all components fail	$F_S(t) = \prod_{i=1}^{N} F_{X_i}(t)$
one component fails	$F_S(t) = 1 - \prod_{i=1}^{N} \left(1 - F_{X_i}(t)\right)$
k components fail, identical distributions	$F_S(t) = \sum_{i=k}^{N} \binom{N}{i} F_X(t)^i \left(1 - F_X(t)\right)^{N-i}$
k components fail, general case	$F_S(t) = \sum_{g \in G_k} \left(\prod_{X \in g} F_X(t) \right) \left(\prod_{X \notin g} \left(1 - F_X(t)\right) \right)$