1. Problem 1

(a) Differentiation with respect to \( t \) would suggest that the m.s. derivative of \( X \) is given by
\[ X'_t = B + 2Ct. \]
Direct verification:
\[
\lim_{h \to 0} E \left[ \left( \frac{X_{t+h} - X_t}{h} - (B + 2Ct) \right)^2 \right] = \lim_{h \to 0} E[h^2C^2] = 0
\]

(b) \( \int_0^1 X_s ds = A + B/2 + C/3 \), which is a \( N(0, \sigma^2) \) random variable with \( \sigma^2 = 1 + \frac{1}{4} + \frac{1}{9} \). So
\[
P \left( \int_0^1 X_s ds \geq 1 \right) = Q \left( \frac{1}{\sigma} \right)
\]

2. Problem 2

(a) Yes. Let \( n \geq 1 \) and \( t_1 < \cdots < t_n < t_{n+1} \). Note that \( Y_{t_{n+1}} = (-1)^D Y_{t_n} \) where \( D = N_{t_{n+1}} - N_{t_n} \). (We write \( D \) for brevity, even though it depends on \( t_n \) and \( t_{n+1} \)). The distribution of \( D \) is Poisson with mean \( \lambda \tau \), where \( \tau = t_{n+1} - t_n \). The vector \( (Y_{t_1}, \ldots, Y_{t_n}) \) is determined by \( (N_t : 0 \leq t \leq t_n) \), and is thus independent of \( D \) by the independent increment property of Poisson processes. To summarize, if \( t_n \) is viewed as the present time, the future value \( Y_{t_{n+1}} \) is a function of the present value \( Y_{t_n} \), and \( D \), which is independent of the past and has a distribution depending on \( (t_n, t_{n+1}) \) only through the difference \( \tau \). Thus, \( Y \) is a time-homogeneous Markov process.

One way to identify the transition rate matrix \( Q \) and the transition probabilities is to directly find the transition probabilities, as follows. For \( i, j \in \{1, -1\} \),
\[
P \left( Y_{t_{n+1}} = j | Y_{t_n} = i \right) = P \left( (-1)^D Y_{t_n} = j | Y_{t_n} = i \right) = P \left( (-1)^D = \frac{i}{j} \right)
\]
Since
\[
P \left( (-1)^D = 1 \right) = \sum_{k=0}^{\infty} \frac{e^{-\lambda \tau} (\lambda \tau)^{2k}}{(2k)!}
\]
\[
= \frac{e^{-\lambda \tau}}{2} \sum_{m=0}^{\infty} \frac{(\lambda \tau)^m + (-\lambda \tau)^m}{m!}
\]
\[
= \frac{e^{-\lambda \tau} (e^{\lambda \tau} + e^{-\lambda \tau})}{2}
\]
\[
= 1 + e^{-2\lambda \tau} \]
it follows that, for \( i, j \in \{1, -1\} \),

\[
p_{ij}(\tau) = \begin{cases} 
\frac{1 + e^{-2\lambda \tau}}{2}, & i = j \\
\frac{1 - e^{-2\lambda \tau}}{2}, & i \neq j
\end{cases}
\]

The transition rates are given by \( q_{ij} = p'_{ij}(0) \), yielding \( Q = \left( \begin{array}{cc} -\lambda & \lambda \\ \lambda & -\lambda \end{array} \right) \).

(b) \( E[|Y_{t+\tau} - Y_t|^2] = 4P(Y_{t+\tau} \neq Y_t) = 2(1 - e^{-2\lambda \tau}) \to 0 \) as \( \tau \to 0 \), so that \( Y \) is m.s. continuous.

(c) If \( Y \) were m.s. differentiable, then \( E[\frac{|Y_{t+\tau} - Y_t|^2}{\tau^2}] \) would have a finite limit as \( \tau \to 0 \). But this expectation is equal to \( \frac{2(1 - e^{-2\lambda \tau})}{\tau^2} \), which diverges to \( \infty \) as \( \tau \to 0 \), so that \( Y \) is not m.s. differentiable.

(d) Note that \( R_Y(t + \tau, t) = E[Y_{t+\tau}Y_t] = P\{Y_{t+\tau} = Y_t\} - P\{Y_{t+\tau} \neq Y_t\} = \frac{1+e^{-2\lambda \tau}}{2} - \frac{1-e^{-2\lambda \tau}}{2} = e^{-2\lambda \tau} \). Then,

\[
E\left[\left(\frac{1}{T}\int_0^T Y_{t}dt\right)^2\right] = \frac{1}{T^2} E\left[\int_0^T Y_{t}dt\int_0^T Y_{s}ds\right]
\]

\[
= \frac{1}{T^2} \int_0^T \int_0^T R_Y(s,t)dsdt
\]

\[
= \frac{1}{\lambda T} + \frac{e^{-2\lambda T} - 1}{2\lambda^2 T^2},
\]

which converges to 0 as \( T \to \infty \). Hence, \( \frac{1}{T}\int_0^T Y_{t}dt \) converges to 0 in m.s. sense.

3. Problem 3

(a) Since \( A_n \to A \), the corollary in Section 2.12 of the notes implies that \( E[AA_n] \to E[A^2] \) and \( E[A_n^2] \to E[A^2] \). Since \( A \) and the \( A_n \) are jointly Gaussian, the hint and the statement just proved imply that:

\[
\]

\[
E[A_n^4] = 3E[A_n^2]^2 \to 3E[A^2]^2
\]

\[
\]


(b) By the assumptions and part (a), \( A_n^2 \xrightarrow{m.s.} A^2 \) and \( B_n^2 \xrightarrow{m.s.} B^2 \). Furthermore, \( A_n + B_n \xrightarrow{m.s.} A + B \) and thus \( (A_n + B_n)^2 \xrightarrow{m.s.} (A + B)^2 \). Then, \( (A_n + B_n)^2 - A_n^2 - B_n^2 \xrightarrow{m.s.} (A + B)^2 - A^2 - B^2 \), which is equivalent to what is to be proved.
(c) Fix any \( t \). Then for \( h \neq 0 \), \( \frac{Y_{t+h}-Y_t}{h} = \frac{X_{t+h} - X_t}{h} = A_h B_h \), where \( A_h \to X_t^\prime \) m.s. and \( B_h \to 2X_t \) m.s. as \( h \to 0 \). Therefore, by part (b), \( \frac{Y_{t+h}-Y_t}{h} = A_h B_h \to 2X_t X_t' \) m.s. That is, if \( X \) is mean zero, Gaussian and m.s. differentiable, then \( X^2 \) is also m.s. differentiable, and, just as for deterministic functions, \( (X^2)' = 2XX' \).

4. Problem 4

(a) A stationary Markov chain is also time homogeneous. Hence, the state transition matrix from \( t \) to \( t+\tau \), \( \tau \geq 0 \), denoted by \( H(\tau) \), can be obtained by solving

\[
\frac{\partial H(\tau)}{\partial \tau} = H(\tau)Q,
\]

and we have

\[
H(\tau) = \begin{bmatrix}
\frac{e^{-2\alpha \tau} + 1}{2} & \frac{1 - e^{-2\alpha \tau}}{2} \\
\frac{1 - e^{-2\alpha \tau}}{2} & \frac{e^{-2\alpha \tau} + 1}{2}
\end{bmatrix}
\]

Then,

\[
R_X(\tau) = \mathbb{E}[X_{t+\tau} X_t]
\]

\[= \mathbb{E}[X_{t+\tau} | X_t = 1]P(X_t = 1) - \mathbb{E}[X_{t+\tau} | X_t = -1]P(X_t = -1) \]

\[= e^{-2\alpha \tau}\]

Hence, the correlation function for general \( \tau \in \mathbb{R} \) is \( R_X(\tau) = e^{-2\alpha |\tau|} \).

(b) For any \( \alpha \geq 0 \), \( R_X(\tau) \) is continuous over \( \mathbb{R} \). Hence, \( X \) is m.s. continuous (Proposition 7.7 on textbook).

(c) When \( \alpha = 0 \), \( R_X'(\tau) = R_X''(\tau) = 0 \), for all \( \tau \in \mathbb{R} \). Then, \( R_X(\tau), R_X'(\tau), R_X''(\tau) \) exist and are continuous in \( \tau \). Hence, \( X \) is m.s. continuously differentiable (Proposition 7.10 on textbook). When \( \alpha > 0 \), \( R_X' \) does not exist at \( \tau = 0 \). Hence, \( X \) is not m.s. continuously differentiable.

(d) When \( \alpha > 0 \), \( \lim_{\tau \to \infty} R_X(\tau) = 0 \), by Proposition 7.18, \( X \) is mean ergodic in the m.s. sense. When \( \alpha = 0 \), \( C_X(\tau) = R_X(\tau) = 1 \) (since the Markov chain has zero mean). Then, \( \lim_{\tau \to \infty} C_X(\tau) = 1 \neq 0 \). Hence, \( X \) is not mean ergodic in the m.s. sense.

5. Problem 5

(a)

\[
R_X(s,t) = \mathbb{E} \left[ \int_0^s Z_u e^{-u} du \int_0^t Z_v e^{-v} dv \right]
\]

\[= \int_0^s \int_0^t \sigma^2 e^{-u-v} \delta(u-v) du dv \]

\[= \int_0^{s \wedge t} \sigma^2 e^{-2v} dv = \frac{\sigma^2}{2} (1 - e^{-2(s \wedge t)})
\]
(b) No. If the ratio \( \frac{X_{t+h} - X_t}{h} \) converged in the m.s. sense as \( h \to 0 \), its mean square value would converge to a finite number. But
\[
\mathbb{E}\left(\left(\frac{X_{t+h} - X_t}{h}\right)^2\right) = \frac{\sigma^2}{h^2} \int_t^{t+h} e^{-2u} du \to \infty,
\]
so the answer is no.

(c) Yes. Observe from part (a) that \( \lim_{t,s \to \infty} R_X(s,t) \) exists and is finite. Thus, the Cauchy criterion for m.s. convergence (Proposition 2.11) holds.

6. Problem 6

(a) The sample paths of \( Y \) are piecewise linear. The initial slope is zero, and the slope of \( Y \) increases by one at each jump of \( N \).

(b)
\[
\mathbb{E}[Y_t] = \int_0^t \mathbb{E}[N_s] ds = \lambda \int_0^t s ds = \frac{\lambda t^2}{2}
\]

(c) Using the fact \( C_N(u,v) = \lambda(u \land v) \), yields
\[
\text{Var}(Y_t) = \int_0^t \int_0^t C_N(u,v) dudv = 2\lambda \int_0^t \int_0^u ududv = \lambda \int_0^t v^2 dv = \frac{\lambda t^3}{3}
\]

(d) Note that if \( N_{t/2} \geq 2 \), then \( Y_t \geq t \). So \( P[Y_t < t] \leq P[N_{t/2} \leq 1] = (1 + \frac{\lambda}{2}) \exp(-\frac{\lambda}{2}) \to 0 \) as \( t \to \infty \).

7. Problem 7

(a) The solution to the differential equation is
\[
X_t = e^{-t}x_0 + \int_0^t e^{-(t-u)} N(u) du
\]
Since \( \mu_N \equiv 0 \) it follows that \( \mu_X(t) = e^{-t}x_0 \). Assume that \( s \leq t \), the covariance function of \( X \) is given by:
\[
C_X(s,t) = \text{Cov}(X_s, X_t) = \text{Cov}\left(\int_0^s e^{-(t-u)} N(u) du, \int_0^s e^{-(s-u)} N(u) du\right)
\]
\[
= \int_0^s \int_0^t e^{-(s-u)} e^{-(t-v)} \sigma^2 \delta(u-v) dudv
\]
\[
= \int_0^s e^{-(s-u)} e^{-(t-u)} \sigma^2 du
\]
\[
= \sigma^2 e^{-s-t} \int_0^s e^{2u} du
\]
\[
= \frac{\sigma^2}{2} (e^{s+t} - e^{-s-t})
\]
By the symmetry of $C_X$, it is given in general by

$$C_X(s, t) = \frac{\sigma^2}{2} \left( e^{-|t-s|} - e^{-t-s} \right)$$

(b) Let $r < s < t$. It must be checked that

$$C_X(r, s)C_X(s, t) = C_X(r, t)C_X(s, s)$$

or $(e^{r-s} - e^{-r-s})(e^{s-t} - e^{-s-t}) = (e^{r-t} - e^{-r-t})(1 - e^{-2s})$ which is easily done.

(c) As $t \to \infty$, $\mu_X(t) \to 0$, $C_X(t + \tau, t) \to \frac{\sigma^2}{2} e^{-|\tau|}$, and $R_X(t + \tau, t) = C_X(t + \tau, t) + \mathbb{E}[X_{t+\tau}]\mathbb{E}[X_t] = C_X(t + \tau, t) + e^{-2t-\tau} x_0^2 \to \frac{\sigma^2}{2} e^{-|\tau|}$.

8. Problem 8

(a) It is easy to see that $R_X'(\tau) = -\tau e^{-\tau^2/2}$ and $R_X''(\tau) = (\tau^2 - 1)e^{-\tau^2/2}$. Since $R_X$, $R_X'$ and $R''(X)$ exist and are continuous, $(X_t)$ is m.s. continuously differentiable (Proposition 7.10 on textbook).

$$\mu_X'(t) = \frac{d}{dt}\mu_X = 0, \quad R_X'(\tau) = -R_X''(\tau) = (1 - \tau^2)e^{-\tau^2/2}$$

Since $R_X(\tau) \to 0$ as $\tau \to \infty$, $X$ is mean ergodic in the m.s. sense, and then

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T X_t dt = \mu_X \text{ m.s.}$$

Consider $\mathbb{E}[(\frac{1}{T} \int_0^T X_t dt - 0)^2]$, we have

$$\mathbb{E} \left[ \left( \frac{1}{T} \int_0^T X_t dt \right)^2 \right] = \frac{1}{T^2} \mathbb{E} \left[ \int_0^T X_t dt \int_0^T X_s ds \right] = \frac{1}{T^2} \int_0^T \int_0^T R_X(s, t) ds dt = \frac{1}{T^2} \int_0^T \int_0^T e^{-\frac{(s-t)^2}{2}} ds dt$$

Since for any fixed $t$,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-t)^2}{2}} dx = 1,$$

we have

$$\frac{1}{T^2} \int_0^T \int_0^T e^{-\frac{(s-t)^2}{2}} ds dt \leq \frac{1}{T^2} \int_0^T \sqrt{2\pi} dt = \frac{\sqrt{2\pi}}{T},$$

which converges to 0 as $T \to \infty$. Hence, $\mu_X = 0$. 
(b) $R_{X'}(\tau) = C_{X'}(\tau) = (1 - \tau^2)e^{-\tau^2/2}$. It is easy to see using L'Hopital's rule that $C_{X'}(\tau) \to 0$ as $\tau \to \infty$. Thus $(X'_t)$ is mean ergodic in m.s. sense.

(c) $X_t$ and $X'_t$ are jointly Gaussian, and we have established that they are zero mean. Furthermore,

\begin{align*}
\Var(X_t) &= R_X(0) = 1, \\
\Var(X'_t) &= R'_{X'}(0) = 1, \\
\Cov(X_t, X'_t) &= R_{X'X}(0) = R'_X(0) = 0.
\end{align*}

Thus, for fixed $t$, $X_t$ and $X'_t$ are independent $\mathcal{N}(0, 1)$ random variables.

(d) $R'_{X'X}(\tau) = R'_X(\tau) = -\tau e^{-\tau^2/2}$.

\[
\mathbb{E}[X_1 | X'_2 = 2] = 0 + \Cov(X_1, X'_2) \Var(X'_2)^{-1}(2 - 0) = R'_{X'X}(1)R'_X(0)^{-1}(2) = -2e^{-1/2}
\]