ECE 534 **Fall 2009**

October 13, 2009

Exam 1

- You have 1.5 hours to complete this exam.
- Don't forget to put your name on the answer booklet.
- You are allowed 1 sheet of handwritten notes (8.5"×11', both sides).
- Calculators laptop computers, PDA's, etc. are not permitted.
- Maximum possible score is 50.
- Neatness counts, especially for partial credit towards incorrect solutions.

1. (16 pts) *Convergence.* In each of the following four parts, you are asked a question about the convergence of a sequence of random variables. If you say yes, provide a proof and the limiting random variable. If you say no, disprove or provide a counterexample.

   (a) Let $A_1, A_2, \ldots$ be a sequence of independent events such that $\mathsf{P}(A_n) \to 1$ as $n \to \infty$. Now define a sequence of random variables $X_n = \mathbb{1}_{A_n}$, $n = 1, 2, \ldots$.
   Does X_n converge in probability as $n \to \infty$?

   (b) Suppose $X_n \overset{m.s.}{\to} X$ as $n \to \infty$ and $\mathsf{E}[X_n^4] < \infty$ for all n.
   Does X_n^2 necessarily converge in mean square as $n \to \infty$?

   (c) Suppose $X \sim \text{Unif}[-1, 1]$ and $X_n = X^n$.
   Does X_n converge almost surely as $n \to \infty$?

   (d) Suppose $X_n \overset{d.}{\to} X$, and a_n is a deterministic sequence such that $a_n \to a$ as $n \to \infty$.
   Does $X_n + a_n$ necessarily converge in distribution as $n \to \infty$? (Hint: Characteristic functions.)

2. (10 pts) *CLT and CB.* Let $X_1, X_2, \ldots$ be i.i.d. Bernoulli random variables, with

$$\mathsf{P}\{X_n = 0\} = \frac{3}{4} \quad \text{and} \quad \mathsf{P}\{X_n = 1\} = \frac{1}{4}$$

   Suppose $S_n = \sum_{i=1}^n X_i$.

   (a) Find $M_X(\theta)$, the moment generating function of X_n.

   (b) Use the Central Limit Theorem to find an approximation for $\mathsf{P}\{S_{100} \geq 50\}$ in terms of the $Q(\cdot)$ function.

   (c) Now use the Chernoff Bound to show that

$$\mathsf{P}\{S_{100} \geq 50\} \leq \left(\frac{4}{3}\right)^{-50}$$

3. (10 pts) *MMSE.* Suppose X, Y have joint pdf

$$f_{X,Y}(x,y) = \begin{cases} 6x & \text{if } x, y \geq 0 \text{ and } x + y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $\mathsf{E}[X|Y]$.

(b) Find the MSE achieved by $\mathsf{E}[X|Y]$, i.e. find the minimum MSE.

(c) Find $\hat{\mathsf{E}}[X|Y]$.

4. (14 pts) *Gaussian MMSE.* Suppose X, Y_1, Y_2 are *zero-mean* jointly Gaussian with covariance

$$\text{Cov}\left(\begin{bmatrix} X \\ Y_1 \\ Y_2 \end{bmatrix}\right) = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

(a) Find $\mathsf{P}\{Y_1 + Y_2 - X \geq 10\}$ in terms of the $Q(\cdot)$ function.

(b) Find $\mathsf{E}[X|Y_1]$ and $\mathsf{E}[X|Y_2]$.

(c) Find $f_{X|Y_1,Y_2}(x|y_1, y_2)$.

(d) Find $\mathsf{P}(\{X \geq 2\}|\{Y_1 + Y_2 = 0\})$ in terms of the $Q(\cdot)$ function.

(e) Let $Z = Y_1^2 + Y_2^2$. Find $\hat{\mathsf{E}}[X|Z]$.
(Hint: For $W \sim \mathcal{N}(0, \sigma^2)$, $\mathsf{E}[W^3] = 0$ and $\mathsf{E}[W^4] = 3\sigma^2$.)