ECE 534 **Spring 2009**

December 1, 2009

Solutions to Exam 2

1. Consider a *Gaussian* WSS process $(X_t : t \in \mathbb{R})$ with autocorrelation function:

$$R_X(\tau) = e^{-\tau^2/2}$$

   (a) Show that (X_t) m.s. differentiable, and find the mean and autocorrelation function of the derivative process (X'_t).

   Ans: It is easy to see that $R'_X(\tau) = -\tau e^{-\tau^2/2}$ and $R''_X(\tau) = (\tau^2 - 1)e^{-\tau^2/2}$. Since R_X, R'_X and $R''(X)$ exist and are continuous, (X_t) is m.s. continuously differentiable.

$$\mu_{X'}(t) = \frac{d}{dt}\mu_X = 0, \qquad R_{X'}(\tau) = -R''_X(\tau) = (1-\tau^2)e^{-\tau^2/2}$$

   Note that since $R_X(\tau) \to 0$ as $\tau \to \infty$, $\mu_X = 0$ as well.

   (b) Determine whether or not (X'_t) is mean ergodic in the m.s. sense.

   Ans: $R_{X'}(\tau) = C_{X'}(\tau) = (1-\tau^2)e^{-\tau^2/2}$. It is easy to see using L'Hopital's rule that $C_{X'}(\tau) \to 0$ as $\tau \to \infty$. Thus (X'_t) is mean ergodic in m.s. sense.

   (c) For any fixed t, find the joint distribution of the random variables X_t and X'_t.

   Ans: X_t and X'_t are jointly Gaussian, and we have established that they are zero mean. Furthermore,

$$\mathrm{Var}(X_t) = R_X(0) = 1, \quad \mathrm{Var}(X'_t) = R'_X(0) = 1, \quad \mathrm{Cov}(X_t, X'_t) = R_{X'X}(0) = R'_X(0) = 0.$$

   Thus, for fixed t, X_t and X'_t are independent $\mathcal{N}(0,1)$ random variables.

   (d) Find $\mathsf{E}[X_1 | X'_2 = 2]$.

   Ans: $R_{X'X}(\tau) = R'_X(\tau) = -\tau e^{-\tau^2/2}$.

$$\mathsf{E}[X_1|X'_2 = 2] = 0 + \mathrm{Cov}(X_1, X'_2)\ \mathrm{Var}(X'_2)^{-1}(2-0) = R_{X'X}(1)R_{X'}(0)^{-1}(2) = -2e^{-1/2}$$

2. (12 pts) Let $(N_t : t \geq 0)$ be a Poisson process with parameter $\lambda = 1$.

   (a) Find $\mathsf{P}(\{N_2 \leq 1\}|\{N_1 \leq 1\})$.

   Ans:

$$\mathsf{P}(\{N_2 \leq 1\}|\{N_1 \leq 1\}) = \frac{\mathsf{P}(\{N_2 \leq 1\}, \{N_1 \leq 1\})}{\mathsf{P}\{N_1 \leq 1\}}$$

   Now $\mathsf{P}\{N_1 \leq 1\} = e^{-1} + e^{-1} = 2e^{-1}$, and using the independent increment property of (N_t),

$$\begin{aligned}\mathsf{P}(\{N_2 \leq 1\}, \{N_1 \leq 1\}) &= \mathsf{P}(\{N_1 = 0\}, \{N_2 - N_1 \leq 1\}) + \mathsf{P}(\{N_1 = 1\}, \{N_2 - N_1 = 0\}) \\ &= e^{-1}(e^{-1} + e^{-1}) + e^{-1}\, e^{-1} = 3e^{-2}\end{aligned}$$

   Thus $\mathsf{P}(\{N_2 \leq 1\}|\{N_1 \leq 1\}) = \frac{3}{2}e^{-1}$

   (b) Find $\mathsf{P}(\{N_1 \leq 1\}|\{N_2 \leq 1\})$.

   Ans: This doesn't need any computation. If $N_2 \leq 1$ then w.p. 1 it must be the case that $N_1 \leq 1$ as well, since (N_t) is non-decreasing w.p. 1. Thus this conditional probability equals 1.

(c) Let $(Y_t : t \geq 0)$ be defined by $Y_t = N_t^2$. Determine whether or not (Y_t) is a Markov process.
Ans: Since (N_t) is non-negative, the mapping from (N_t) to (Y_t) is one-one. Thus (Y_t) inherits the Markov property from (N_t).

(d) Determine whether or not (Y_t) has independent increments.
Ans: No, (Y_t) is not an independent increment process. Consider,

$$\begin{aligned}\mathsf{E}[(Y_2 - Y_1)|(Y_1 - Y_0)] &= \mathsf{E}[(N_2^2 - N_1^2)|N_1] = \mathsf{E}[(N_2 - N_1)^2 + 2N_1N_2 - 2N_1^2|N_1] \\ &= \mathsf{E}[(N_2 - N_1)^2|N_1] + 2N_1\mathsf{E}[(N_2 - N_1)|N_1] \\ &= \mathsf{E}[(N_2 - N_1)^2] + 2N_1\mathsf{E}[(N_2 - N_1)] = 2 + 2N_1\end{aligned}$$

The RHS is a function of N_1, i.e., it is a function of Y_1, and hence the increment $(Y_2 - Y_1)$ is not independent of the increment $(Y_1 - Y_0)$.

3. (12 pts) Let $(W_t : t \geq 0)$ be a Brownian motion with parameter $\sigma^2 = 1$.

(a) Find $\mathsf{P}(\{W_1 + W_2 \geq 2\}|\{W_1 = 1\})$.
Ans: Since W_1, W_2 are zero mean and jointly Gaussian,

$$\begin{aligned}\mathsf{P}(\{W_1 + W_2 \geq 2\}|\{W_1 = 1\}) &= \mathsf{P}(\{2W_1 + (W_2 - W_1) \geq 2\}|\{W_1 = 1\}) \\ &= \mathsf{P}\{(W_2 - W_1) \geq 0\} = Q(0) = \frac{1}{2}\end{aligned}$$

(b) Now suppose we define the random variable Z via the m.s. integral

$$Z = \int_0^1 W_t dt$$

Find $\mathsf{E}[W_2|Z]$.
Ans: Z is zero-mean Gaussian random variable with

$$\mathrm{Var}(Z) = \mathsf{E}[Z^2] = \int_0^1 \int_0^1 \min(s,t)\, dtds = 2\int_{t=0}^1 \int_{s=0}^t = \frac{1}{3}$$

Also,

$$\mathrm{Cov}(W_2, Z) = \mathsf{E}[W_2 Z] = \int_0^1 \mathsf{E}[W_2 W_t]dt = \int_0^1 tdt = \frac{1}{2}$$

Thus $\mathsf{E}[W_2|Z] = 0 + \mathrm{Cov}(W_2, Z)\mathrm{Var}(Z)^{-1}(Z - 0) = \frac{3}{2}Z$

(c) Let Let $(Y_t : t \geq 0)$ be defined $Y_t = W_t^3$. Determine whether or not (Y_t) is a martingale.
Ans: If (Y_t) was a martingale, then we would have $\mathsf{E}[Y_2|Y_1] = Y_1$. But consider

$$\begin{aligned}\mathsf{E}[Y_2|Y_1] &= \mathsf{E}[W_2^3|W_1^3] = \mathsf{E}[W_2^3|W_1] = \mathsf{E}[(W_2 - W_1 + W_1)^3|W_1] \\ &= \mathsf{E}[(W_2 - W_1)^3 + 3(W_2 - W_1)W_1^2 + 3(W_2 - W_1)^2 W_1 + W_1^3|W_1] \\ &= 0 + 0 + 3W_1\mathsf{E}[(W_2 - W_1)^2] + W_1^3 = 3W_1 + W_1^3 \neq Y_1\end{aligned}$$

Thus (Y_t) is not a martingale.

4. (14 pts, equally weighted parts) *True or False.*

   (a) Let $\Theta \sim \text{Unif}[0, 2\pi]$ and let the random process $(X_t : t \geq 0)$ be defined by $X_t = \cos(2\pi t + \Theta)$. Then (X_t) is an independent increment process.

   Ans: False. $X_1 - X_{0.5} = \cos(2\pi_\Theta) - \cos(\pi + \Theta) = 2\cos(\Theta)$ and $X_2 - X_{1.5} = \cos(4\pi_\Theta) - \cos(3\pi + \Theta) = 2\cos(\Theta)$ as well. These two increments are equal and hence certainly not independent.

   (b) If (X_t) and (Y_t) are *independent* WSS Gaussian processes, then the process (Z_t) defined by $Z_t = X_t Y_t$ is also WSS.

   Ans: True. $\mathsf{E}[Z_t] = \mathsf{E}[X_t]\mathsf{E}[Y_t] = \mu_X \mu_Y$, which is not a function of t, and $\mathsf{E}[Z_t Z_{t+\tau}] = \mathsf{E}[X_t X_{t+\tau}]\mathsf{E}[Y_t Y_{t+\tau}] = R_X(\tau)R_Y(\tau)$, which is only a function of τ.

   (c) If $(X_t, t \geq 0)$ is a martingale, then (X_t) is necessarily WSS as well.

   Ans: False. The Weiner process $(W_t, t \geq 0)$ is a martingale but it is not WSS.

   (d) Consider a WSS process (X_t) with autocorrelation function $R_X(\tau)$ that converges to 2 as $\tau \to \infty$. Then (X_t) cannot be mean square ergodic in the m.s. sense.

   Ans: False. Consider (X_t) which is WSS with mean $\mu_X = \sqrt{2}$ and auto-covariance function $C_X(\tau) = e^{-|\tau|}$. Then (X_t) is mean ergodic in m.s. sense and $R_X(\tau)$ that converges to 2 as $\tau \to \infty$.

   (e) If $(X_t : t \in \mathbb{R})$ is a (continuous-time) stationary Gauss-Markov process, then the samples $(X_k, k \in \mathbb{Z})$ form a discrete-time stationary Gauss-Markov process.

   Ans: True. The continuous-time process (X_t) has auto-covariance function $C_X(\tau) = \sigma^2 e^{-\alpha|\tau|}$ for some $\alpha > 0$. The sample process is obviously Gaussian and has auto-covariance function

$$C_X(k, k+m) = \mathsf{E}[X_k X_{k+m}] = \sigma^2 e^{-\alpha|m|} = \sigma^2 \left(e^{-\alpha}\right)^{|m|}$$

   Thus $C_X(k, k+m)$ has the necessary form for a discrete-time stationary Gauss-Markov process.

   (f) If $(X_t : t \in \mathbb{R})$ is a zero-mean WSS Gaussian process, then the process (Y_t) defined by $Y_t = X_t^2$ is also a WSS process.

   Ans: True. The mean $\mathsf{E}[Y_t] = \mathsf{E}[X_t^2] = R_X(0)$ is independent of t. Also, using the hint,

$$R_Y(t, t+\tau) = \mathsf{E}[Y_t Y_{t+\tau}] = \mathsf{E}[X_t X_t X_{t+\tau} X_{t+\tau}] = \mathsf{E}[X_t^2]\mathsf{E}[X_{t+\tau}^2] + 2(\mathsf{E}[X_t X_{t+\tau}])^2 = R_X(0)^2 + 2R_X(\tau)^2$$

   which is independent of τ. Thus (Y_t) is WSS.

   Alternatively, you could argue that since (X_t) is not just WSS but also stationary, (Y_t) must also be stationary. Thus (Y_t) must be WSS as well.

   (g) Consider the function $R(\tau)$ defined by:

$$R(\tau) = \begin{cases} 1 + (1 - |\tau|) & \text{if } |\tau| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   $R(\tau)$ is valid autocorrelation function for a WSS process.

   Ans: False. It is easy to see that $R(\tau)$ is continuous at $\tau = 0$ but not at $\tau = +1$ or at $\tau = -1$.