November 17, 2008

Exam 2

- You have 75 minutes to complete this exam.
- Don't forget to put your name on the answer booklet.
- You are allowed 2 sheets of notes (8.5"×11', both sides).
- Calculators laptop computers, PDA's, etc. are not permitted.
- Maximum possible score is 50.
- Neatness counts, especially for partial credit towards incorrect solutions.

1. (12 pts) Consider a *Gaussian* WSS process $(X_t : t \in \mathbb{R})$ with autocorrelation function:

$$R_X(\tau) = e^{-\tau^2/2}$$

   (a) Show that (X_t) m.s. differentiable, and find the mean and autocorrelation function of the derivative process (X'_t).
   (b) Determine whether or not (X'_t) is mean ergodic in the m.s. sense.
   (c) For any fixed t, find the joint distribution of the random variables X_t and X'_t.
   (d) Find $\mathsf{E}[X_1 | X'_2 = 2]$.

2. (12 pts) Let $(N_t : t \geq 0)$ be a Poisson process with parameter $\lambda = 1$.

   (a) Find $\mathsf{P}(\{N_2 \leq 1\} | \{N_1 \leq 1\})$.
   (b) Find $\mathsf{P}(\{N_1 \leq 1\} | \{N_2 \leq 1\})$.
   (c) Let $(Y_t : t \geq 0)$ be defined by $Y_t = N_t^2$. Determine whether or not (Y_t) is a Markov process.
   (d) Determine whether or not (Y_t) has independent increments.

3. (12 pts) Let $(W_t : t \geq 0)$ be a Brownian motion with parameter $\sigma^2 = 1$.

   (a) Find $\mathsf{P}(\{W_1 + W_2 \geq 2\} | \{W_1 = 1\})$.
   (b) Now suppose we define the random variable Z via the m.s. integral

$$Z = \int_0^1 W_t dt$$

   Find $\mathsf{E}[W_2 | Z]$.
   (c) Let Let $(Y_t : t \geq 0)$ be defined $Y_t = W_t^3$. Determine whether or not (Y_t) is a martingale.

4. (14 pts, equally weighted parts) *True or False.* Determine if the following statements are True or False. You need to justify your answer clearly to get credit – provide a short proof if you say the statement is True, and a counter-example if you say the statement is False. Just stating "True" or "False" without any justification will get zero credit.

   You may use without proof the following fact: if X_1, X_2, X_3, X_4 are jointly Gaussian (correlated) random variables with *zero mean*, then

$$\mathsf{E}[X_1X_2X_3X_4] = \mathsf{E}[X_1X_2]\mathsf{E}[X_3X_4] + \mathsf{E}[X_1X_3]\mathsf{E}[X_2X_4] + \mathsf{E}[X_1X_4]\mathsf{E}[X_2X_3]$$

   (a) Let $\Theta \sim \text{Unif}[0, 2\pi]$ and let the random process $(X_t : t \geq 0)$ be defined by $X_t = \cos(2\pi t + \Theta)$. Then (X_t) is an independent increment process.

   (b) If (X_t) and (Y_t) are *independent* WSS Gaussian processes, then the process (Z_t) defined by $Z_t = X_tY_t$ is also WSS.

   (c) If $(X_t, t \geq 0)$ is a martingale, then (X_t) is necessarily WSS as well.

   (d) Consider a WSS process (X_t) with autocorrelation function $R_X(\tau)$ that converges to 2 as $\tau \to \infty$. Then (X_t) cannot be mean square ergodic in the m.s. sense.

   (e) If $(X_t : t \in \mathbb{R})$ is a (continuous-time) stationary Gauss-Markov process, then the samples $(X_k, k \in \mathbb{Z})$ form a discrete-time stationary Gauss-Markov process.

   (f) If $(X_t : t \in \mathbb{R})$ is a zero-mean WSS Gaussian process, then the process (Y_t) defined by $Y_t = X_t^2$ is also a WSS process.

   (g) Consider the function $R(\tau)$ defined by:

$$R(\tau) = \begin{cases} 1 + (1 - |\tau|) & \text{if } |\tau| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   $R(\tau)$ is valid autocorrelation function for a WSS process.