ECE 534 Random Processes *Instructor: R. Srikant*

Midterm Exam II

Spring '07 **Time: 7:00-8:30 pm**

You are allowed one $8.5'' \times 11''$ **sheet (two pages) of handwritten notes.**

No calculators or electronic devices are allowed.

Each problem is worth 20 points. There are five problems.

Please write your name here: ______________________________

1. Let $X = \{X_t,\, t \in \mathcal{R}\}$ be a WSS Gaussian process.

(a) Let $Y_t = X_t + X_{t-2}$. Is $Y = \{Y_t, t \in \mathcal{R}\}$ a WSS process? Is it a stationary process? Clearly explain your answer.

X is WSS and Gaussian. This means X is stationary and Gaussian. Now obviously $E(Y_t) = 2\mu_X$ and

$E(Y_{t_1}Y_{t_2}) = E((X_{t_1} + X_{t_1-2})(X_{t_2} + X_{t_2-2})) = 2R_X(t_1 - t_2) + R_X(t_1 - t_2 + 2) + R_X(t_1 - t_2 - 2) = R_Y(t_1 - t_2)$.

Since $R_Y(t_1, t_2)$ depends only on $t_1 - t_2$, Y is WSS too. Since X is Gaussian and Y is a linear function of X, Y is also Gaussian. Thus, Y is also stationary.

(b) What is the necessary and sufficient condition on R_X for X to be *Markov* process?

Given it is Gaussian and stationary, it will be Markov if and only if $R_X(\tau) = A\ \exp(-\alpha|\tau|) + \mu_X^2$ for some constants $A > 0$ and $\alpha \geq 0$.

(c) If X is *Markov*, is Y also *Markov* ? Clearly explain your answer.

Let $R_X(\tau)$ be given by the expression in part (b). Now, substituting this in the formula for $R_Y(\tau)$ from part (a), we can immediately see that R_Y cannot be in the required form so Y is not *Markov*.

2. A two-dimensional spatial Poisson process is used to model the distribution of points on a two-dimensional plane. For example, it could be used to model the distribution of wireless communication devices in Champaign-Urbana. Let N be a two-dimensional Poisson process of intensity λ. N is defined as follows:

(i) For any set $B \in \mathcal{R}^2$, the number of points in B, denoted by N_B, is a Poisson random variable with mean equal to $\lambda *$Area(B).

(ii) If $S_1, S_2, \ldots, S_n$ are disjoint sets in $\mathcal{R}^2$, then $N_{S_1}, N_{S_2}, \ldots, N_{S_n}$ are mutually independent.

Answer the following questions:

(a) What is the probability that there are no points in a circle of radius r centered at the origin?

Since the area of the circle is $r^2\pi$, the probability of the event is equal to $e^{-\lambda r^2 \pi}$.

(b) Let R be the distance from the origin of the closest point to the origin. Find the *cdf* and *pdf* of R.

$F(r) = P(R \leq r) = 1 - P(R > r) = 1 - e^{-\lambda r^2 \pi}$, so $f(r) = 2\lambda\pi r e^{-\lambda r^2 \pi}$.

3. Let W be a standard Brownian motion, i.e, a Brownian motion with $\sigma^2 = 1$. Define $X_t = e^{-t} W_{e^{2t}}$.

(a) Find the mean and autocorrelation function of X.

$E(X_t) = 0$ and, if $t_1 \leq t_2$, $E(X_{t_1} X_{t_2}) = e^{-(t_1+t_2)} E(W_{e^{2t_1}} W_{e^{2t_2}}) = e^{-(t_1+t_2)} e^{2t_1} = e^{t_1 - t_2}$. Thus, X is WSS with $R_X(\tau) = e^{-|\tau|}$.

(b) Is X mean ergodic in the m.s. sense?

$C_X(\tau) = R_X(\tau) - \mu_X^2 = e^{-|\tau|}$, so $\lim_{\tau \to \infty} C_X(\tau) = 0$ and thus, X is mean ergodic m.s.

4. $X = \{X_t, t \in \mathcal{R}\}$ is WSS with $\mu_X = 0$ and $R_X(t) = e^{-2|t|}$.

The process Y is defined by the stochastic differential equation:

$$Y_t' = -2Y_t + X_t, \qquad Y_0 = 1.$$

Find $\mu_Y(t)$ and $R_{XY}(t_1, t_2)$.

From given equation we deduce that $Y_t = Ke^{-2t} + e^{-2t} \int_0^t X_\tau e^{2\tau} d\tau$ and given $Y_0 = 1$, we have $K = 1$. So $\mu_Y(t) = e^{-2t} + e^{-2t} \int_0^t \mu_X(\tau) e^{2\tau} d\tau = e^{-2t}$. Similarly, $R_{XY}(t_1, t_2) = e^{-2t_2} \int_0^{t_2} R_X(t_1 - \tau) e^{2\tau} d\tau$ and thus,

$$R_{XY}(t_1, t_2) = \begin{cases} \frac{1}{4}(e^{2(t_2 - t_1)} - e^{-2t_1 - 2t_2}) & \text{if } t_2 \leq t_1; \\ e^{-2t_2}[\frac{1}{4}(e^{2t_1} - e^{-2t_1}) + e^{2t_1}(t_2 - t_1)] & \text{if } t_1 \leq t_2 \end{cases}$$

5. Let N_n be the number of heads in n tosses of a biased coin with $Prob(\text{head}) = p$.
What is $\lim_{n \to \infty} P(N_n$ is a multiple of 3)? Clearly justify your answer.
Hint: Define an appropriate Markov chain and find its stationary distribution. You can assume that, starting from any initial distribution, the distribution of the Markov chain converges to the stationary distribution.

Consider the Markov chain $X_n = N_n$ mod 3, i.e., the remainder of N_n when divided by 3. From the "Hint", π_0 is the required answer. The probability transition matrix of X_n is given by:

$$P = \begin{pmatrix} 1-p & p & 0 \\ 0 & 1-p & p \\ p & 0 & 1-p \end{pmatrix}$$

Solving the equation $\pi P = \pi$, or by symmetry, we can conclude that $\pi_0 = 1/3$.