Solutions to Midterm Exam I

(1)

(1) (a)

$$\psi_X(u) = E\left(e^{jux}\right) = \int_0^1 e^{jux}\,dx = \left.\frac{e^{jux}}{ju}\right|_0^1 = \frac{e^{ju}-1}{ju}$$

(b), (c)

$$\psi_{X_n}(u) = E\left(e^{ju\sum_{i=1}^{n}\frac{X_i}{10^i}}\right)$$

$$= \prod_{i=1}^{n} E\left(e^{ju\frac{X}{10^i}}\right)$$

$$= \prod_{i=1}^{n}\left[\sum_{k=0}^{9}\frac{e^{ju\frac{k}{10^i}}}{10}\right]$$

$$= \frac{1}{10^n}\prod_{i=1}^{n}\frac{1-e^{\frac{ju}{10^{i-1}}}}{1-e^{\frac{ju}{10^i}}}$$

$$= \frac{1}{10^n}\,\frac{\left(1-e^{\frac{ju}{10^0}}\right)}{1-e^{\frac{ju}{10^n}}}$$

(d)

$$\simeq \frac{1}{10^n}\,\frac{1-e^{ju}}{-\frac{ju}{10^n}} = \frac{e^{ju}-1}{ju} \text{ as } n\to\infty.$$

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$X_n \xrightarrow{d} \text{Unif}[0,1]$

(1) $X_n \le X_{n+1} \le 1$

$\Rightarrow$ X_n converges for each $\omega \in \Omega$

$\Rightarrow$ X_n converges a.s. But a.s. $\Rightarrow$ d. d

we have already seen that the limit has

Unif[0,1] distribution.

(2)

$$Y_i = \begin{cases} 1 & \text{if } i^{th} \text{ toss is } H \\ -1 & \text{if } \quad \quad T \end{cases}$$

$$H_n - T_n = \sum_{i=1}^{n} Y_i$$

$$M(\theta) = \ln E(e^{\theta Y_i}) = \ln\left[\frac{e^{\theta} + e^{-\theta}}{2}\right]$$

$$l(x) = \max_{\theta} \; \theta x - \ln\left[\frac{e^{\theta} + e^{-\theta}}{2}\right]$$

$$\Rightarrow \quad x - \frac{1}{\left(\frac{e^{\theta}+e^{-\theta}}{2}\right)} \cdot \frac{(e^{\theta} - e^{-\theta})}{2} = 0$$

$$e^{\theta}(x-1) + e^{-\theta}(x+1) = 0$$

$$e^{2\theta} = \frac{x+1}{1-x} \quad \text{or} \quad e^{\theta} = \sqrt{\frac{x+1}{1-x}}$$

$$l(x) = \ln\left[\frac{2e^{\theta x}}{e^{\theta}+e^{-\theta}}\right]$$

$$= \ln\left[\frac{2}{e^{\theta(1-x)}+e^{-\theta(1+x)}}\right]$$

$$= \ln\left[\frac{2}{\left(\sqrt{\frac{x+1}{1-x}}\right)^{1-x}+\left(\sqrt{\frac{1-x}{1+x}}\right)^{1+x}}\right]$$

$$= \ln\left[\frac{2\left(\sqrt{1+x}\right)^{1+x}\left(\sqrt{1-x}\right)^{1-x}}{(1+x)+(1-x)}\right]$$

$$= \ln\left[\frac{\cancel{2}\left(\sqrt{1+x}\right)^{1+x}\left(\sqrt{1-x}\right)^{1-x}}{\cancel{2}}\right]$$

$$\Rightarrow \quad e^{n\,l(x)} = \left(\frac{1}{\left(\sqrt{1+x}\right)^{1+x}\left(\sqrt{1-x}\right)^{1-x}}\right)^{n}$$

3. (a) $\hat{X} = \sum_{i=1}^{n} a_i Y_i$ (note: $E(X) = E(W_i) = 0$)

By symmetry,

$$\hat{X} = a\left(\sum_{i=1}^{n} Y_i\right)$$

By the orthogonality principle,

$$E\left(\left(X - a\sum_{i=1}^{n} Y_i\right) Y_1\right) = 0$$

$$E(XY_1) = E(X(X+W_1)) = 1$$

$$E(Y_i Y_1) = E((X+W_i)(X+W_1)) = E(X^2) + E(W_1 W_i) = \begin{cases} 1+\sigma^2 & \text{if } i=1 \\ 1 & \text{else} \end{cases}$$

$$\Rightarrow \boxed{a = \frac{1}{n+\sigma^2}}$$

(b) $E(X^2) = 1$

$$E(\hat{X}^2) = E\left(a^2\left(\sum_{i=1}^{n} Y_i\right)^2\right) = \frac{1}{(n+\sigma^2)^2}\, E\left(\left[\sum_{i=1}^{n}(X+W_i)\right]^2\right)$$

$$E(X+W_i)^2 = 1+\sigma^2$$

$$E((X+W_i)(X+W_j)) = 1$$

F

$$E(\hat{x}^2) = n(1+\sigma^2) + n(n-1)$$

$$= n\sigma^2 + n^2 = n(n+\sigma^2)$$

$$\Rightarrow E(\hat{x}^2) = \frac{n}{n+\sigma^2}$$

$$Var(e) = E(x^2) - E(\hat{x}^2) = 1 - \frac{n}{n+\sigma^2} = \frac{\sigma^2}{n+\sigma^2}.$$

(4) $$y = \hat{E}(y|y) = \hat{E}(x|y) + \hat{E}(w|y)$$ (since $Cov(w,y) = Cov(x,y)$)

By symmetry, $$\hat{E}(x|y) = \hat{E}(w|y)$$

$$\Rightarrow \boxed{\hat{E}(x|y) = \frac{y}{2}}$$ does not depend on independence or not of X & W.

(5) Given x, Y is Unif$[x^2, x^2+1]$

$$E(Y|x) = x^2 + 0.5$$

Since $E(Y|x)$ is quadratic, the answer to part (b) is also $x^2+0.5$.