Solutions to Exam 1

1. Convergence. In each of the following four parts, you are asked a question about the convergence of a sequence of random variables. If you say yes, provide a proof and the limiting random variable. If you say no, disprove or provide a counterexample.

(a) Let \( A_1, A_2, \ldots \) be a sequence of independent events such that \( P(A_n) \to 1 \) as \( n \to \infty \). Now define a sequence of random variables \( X_n = \mathbb{1}_{A_n}, n = 1, 2, \ldots \). Does \( X_n \) converge in probability as \( n \to \infty \)?

\[ \text{Ans:} \text{ No. Consider } \Omega = [0,1] \text{ with the uniform probability measure, and let } X_n = n \mathbb{1}_{[0,1/n^4]} \]. Then \( E[X^4] = 1 < \infty \) for all \( n \), and \( X_n \xrightarrow{m.s.} X \), with \( X = 0 \) a.s., but \( E[X^2_n X^2_{n-1}] = n^2(n-1)^2/n^4 \to 1 \neq E[X^2 X^2] = 0 \). Thus, by the Cauchy criterion, \( X_n^2 \) does not converge in m.s. sense.

(b) Suppose \( X_n \xrightarrow{m.s.} X \) as \( n \to \infty \) and \( E[X^4_n] < \infty \) for all \( n \). Does \( X_n^2 \) necessarily converge in mean square as \( n \to \infty \)?

\[ \text{Ans:} \text{ No. Consider } \Omega = [0,1] \text{ with the uniform probability measure, and let } X_n = n \mathbb{1}_{[0,1/n^4]} \]. Then \( E[X^4] = 1 < \infty \) for all \( n \), and \( X_n \xrightarrow{m.s.} X \), with \( X = 0 \) a.s., but \( E[X^2_n X^2_{n-1}] = n^2(n-1)^2/n^4 \to 1 \neq E[X^2 X^2] = 0 \). Thus, by the Cauchy criterion, \( X_n^2 \) does not converge in m.s. sense.

(c) Suppose \( X \sim \text{Unif}[-1,1] \) and \( X_n = X^n \). Does \( X_n \) converge almost surely as \( n \to \infty \)?

\[ \text{Ans:} \text{ Yes. } X_n(\omega) = X(\omega)^n \to 0 \text{ for all } \omega \text{ except that for which } X(\omega) = 1 \text{ or } X(\omega) = -1, \text{ which belong to set of measure } 0. \text{ Thus } X_n \xrightarrow{a.s.} 0. \]

(d) Suppose \( X_n \xrightarrow{d} X \), and \( a_n \) is a deterministic sequence such that \( a_n \to a \) as \( n \to \infty \). Does \( X_n + a_n \) necessarily converge in distribution as \( n \to \infty \)?

\[ \text{Ans:} \text{ Yes. Using characteristic functions, we have } E[e^{juX_n}] \to E[e^{juX}] \text{ for all } u \in \mathbb{R}. \text{ Thus } E[e^{j(X_n+a_n)u}] = e^{jau} E[e^{jX_nu}] \to e^{jau} E[e^{jXu}] = E[e^{j(X+a)u}] \]

which means that \( X_n + a_n \xrightarrow{d} X + a. \)

2. Let \( X_1, X_2, \ldots \) be i.i.d. Bernoulli random variables, with

\[ P\{X_n = 0\} = \frac{3}{4} \text{ and } P\{X_n = 1\} = \frac{1}{4} \]

Suppose \( S_n = \sum_{i=1}^{n} X_i \).

(a) Find \( M_X(\theta) \), the moment generating function of \( X_n \).

\[ \text{Ans:} \text{ } M_X(\theta) = E[e^{\theta X_n}] = \frac{1}{4} e^\theta + \frac{3}{4}. \]

(b) Use the Central Limit Theorem to find an approximation for \( P\{S_{100} \geq 50\} \) in terms of the \( Q(\cdot) \) function.

\[ \text{Ans:} \mu = E[X_n] = \frac{1}{4} \text{ and } \sigma^2 = \text{Var}(X_n) = E[X_n^2] - \mu^2 = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}. \text{ Thus, by the Central Limit Theorem, } (S_{100} - 100\mu)/(10\sigma) \text{ is approximately } \mathcal{N}(0,1). \text{ Therefore, } \]

\[ P\{S_{100} > 50\} = P\left\{ \frac{S_{100} - 100\mu}{10\sigma} > \frac{50 - n\mu}{10\sigma} \right\} \approx Q\left( \frac{50 - n\mu}{10\sigma} \right) = Q\left( \frac{10}{\sqrt{3}} \right) \]
(c) Now use the Chernoff Bound to show that

\[ P\{S_{100} \geq 50\} \leq \left(\frac{4}{3}\right)^{-50} \]

**Ans:** By the Chernoff Bound,

\[ P\{S_{100} \geq 50\} = P\left\{\frac{S_{100}}{100} \geq \frac{1}{2}\right\} \leq e^{-100 \ell(0.5)} \]

where \( \ell(0.5) \) is obtained by maximizing

\[ 0.5 \theta - \ln M_X(\theta) = 0.5 \theta - \ln(3 + e^\theta) + \ln(4) \]

Taking the derivative and setting it equal to zero, we obtain that the optimizing \( \theta^* \) satisfies

\[ 0.5 = \frac{e^{\theta^*}}{3 + e^{\theta^*}} \implies \theta^* = \ln 3 \]

Thus \( \ell(0.5) = 0.5 \ln 3 - \ln(3/2) = 0.5 \ln 4 - 0.5 \ln 3 \), and the upper bound follows.

3. (12 pts) Suppose \( X, Y \) have joint pdf

\[ f_{X,Y}(x, y) = \begin{cases} 6x & \text{if } x, y \geq 0 \text{ and } x + y \leq 1 \\ 0 & \text{otherwise} \end{cases} \]

(a) Find \( E[X|Y] \).

**Ans:** \( f_{X,Y}(x, y) = 6x \ \mathbb{1}_{(0 \leq y \leq 1)} \ \mathbb{1}_{(0 \leq x \leq 1-y)} \). Thus

\[ f_y(y) = \int_0^{1-y} 6xdx \ \mathbb{1}_{(0 \leq y \leq 1)} = 3(1-y)^2 \ \mathbb{1}_{(0 \leq y \leq 1)} \]

and for \( 0 \leq y \leq 1 \),

\[ f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_y(y)} = \frac{2x}{(1-y)^2} \ \mathbb{1}_{(0 \leq x \leq 1-y)} \]

Therefore, for \( 0 \leq y \leq 1 \),

\[ E[X|Y = y] = \int_0^{1-y} x f_{X|Y}(x|y)dx = \frac{2}{3} \frac{(1-y)^3}{(1-y)^2} = \frac{2}{3}(1-y) \]

and \( E[X|Y] = \frac{2}{3}(1 - Y) \).

(b) Find the MSE achieved by \( E[X|Y] \), i.e. find the minimum MSE.

**Ans:** It is easy to see that \( f_x(x) = 6x(1-x) \mathbb{1}_{(0 \leq x \leq 1)} \). Thus, the minimum MSE is given by

\[ \text{MSE} = E[X^2] - E[(E[X|Y])^2] = \int_0^1 6x^3(1-x)dx - \frac{4}{9} \int_0^1 3(1-y)^4dy = \frac{3}{10} - \frac{4}{15} = \frac{1}{30} \]

(c) Find \( \hat{E}[X|Y] \).

**Ans:** Since \( E[X|Y] \) is linear in \( Y \), \( \hat{E}[X|Y] = E[X|Y] \).
4. (14 pts) Suppose $X, Y_1, Y_2$ are zero-mean jointly Gaussian with covariance matrix

$$\text{Cov} \left( \begin{bmatrix} X \\ Y_1 \\ Y_2 \end{bmatrix} \right) = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

(a) Find $P\{Y_1 + Y_2 - X \geq 10\}$ in terms of the $Q(\cdot)$ function.

Ans: Let $W = Y_1 + Y_2 - X$. Then $W$ is Gaussian with $E[W] = 0$ and


Thus $P\{Y_1 + Y_2 - X \geq 10\} = Q(\sqrt{10})$.

(b) Find $E[X|Y_1]$ and $E[X|Y_2]$.

Ans: $E[X|Y_1] = 0 + \text{Cov}(X, Y_1)\text{Cov}(Y_1)^{-1}(Y - 0) = -Y_1$. Similarly, $E[X|Y_2] = -Y_2$.

(c) Find $f_{X|Y_1, Y_2}(x|y_1, y_2)$.

Ans: We know that given $Y_1 = y_1, Y_2 = y_2, X$ is Gaussian with mean $E[X|Y_1 = y_1, Y_2 = y_2]$, and variance equal to $\text{Cov}(e)$, with $e = X - E[X|Y_1, Y_2]$. Now, with $\underline{Y} = [Y_1 Y_2]^\top$, 

$$E[X|\underline{Y} = \underline{y}] = 0 + \text{Cov}(X, \underline{Y})\text{Cov}(\underline{Y})^{-1}[\underline{y}] = [-1 - 1][\underline{y} = -y_1 - y_2].$$

(Note: we could have concluded this from part (b) using linear innovations.) Similarly,

$$\text{Cov}(e) = \text{Cov}(X) - \text{Cov}(X, \underline{Y})\text{Cov}(\underline{Y})^{-1}\text{Cov}(\underline{Y}, X) = 4 - [-1 - 1][-1 - 1]^\top = 2$$

Thus $f_{X|Y_1, Y_2}(x|y_1, y_2) \sim N(-y_1 - y_2, 2)$.

(d) Find $P(\{X \geq 2\}|\{Y_1 + Y_2 = 0\})$ in terms of the $Q(\cdot)$ function.

Ans: The straightforward way to do this problem is to define $V = Y_1 + Y_2$, note that $X$ and $V$ are jointly Gaussian, find the conditional distribution of $X$ given $V$ using the MMSE approach, and then compute the above probability. But based on the result of part (c), we can conclude that $f_{X|V}(x|v) \sim N(-v, 2)$. Thus $P(\{X \geq 2\}|\{Y_1 + Y_2 = 0\}) = P(\{X \geq 2\}|\{V = 0\}) = Q(\sqrt{2})$.

(e) Let $Z = Y_1^2 + Y_2^2$. Find $E[X|Z]$.

Ans: Note that $\text{Cov}(X, Z) = E[XY_1^2] + E[XY_2^2] = 0$, since for $i = 1, 2$,

$$E[XY_i^2] = E[E[XY_i^2|Y_i]] = E[Y_i^2 E[X|Y_i]] = -E[Y_i^3] = 0$$

Thus 

$$E[X|Z] = E[X] - 0 = 0$$