ECE 534 **Fall 2009**

October 13, 2009

Solutions to Exam 1

1. *Convergence.* In each of the following four parts, you are asked a question about the convergence of a sequence of random variables. If you say yes, provide a proof and the limiting random variable. If you say no, disprove or provide a counterexample.

   (a) Let $A_1, A_2, \ldots$ be a sequence of independent events such that $\mathsf{P}(A_n) \to 1$ as $n \to \infty$. Now define a sequence of random variables $X_n = \mathbb{1}_{A_n}$, $n = 1, 2, \ldots$. Does X_n converge in probability as $n \to \infty$?

   Ans: We can guess that $X_n \xrightarrow{p.} 1$. To prove this, consider $\mathsf{P}\{|X_n - 1| \geq \epsilon\}$. Clearly $\mathsf{P}\{|X_n - 1| \geq \epsilon\} = 0 \; \forall n$ if $\epsilon > 1$, since $|X_n - 1|$ cannot exceed 1. Thus it remains to see if this probability converges to 0 for $0 < \epsilon \leq 1$. For $0 < \epsilon \leq 1$

   $$\mathsf{P}\{|X_n - 1| \geq \epsilon\} = \mathsf{P}(A_n^c) = 1 - \mathsf{P}(A_n) \to 0 \quad \text{as } n \to \infty$$

   (b) Suppose $X_n \xrightarrow{m.s.} X$ as $n \to \infty$ and $\mathsf{E}[X_n^4] < \infty$ for all n. Does X_n^2 necessarily converge in mean square as $n \to \infty$?

   Ans: No. Consider $\Omega = [0,1]$ with the uniform probability measure, and let $X_n = n\mathbb{1}_{\{\omega \in [0,1/n^4]\}}$. Then $\mathsf{E}[X^4] = 1 < \infty$ for all n, and $X_n \xrightarrow{m.s.} X$, with $X = 0$ a.s., but $\mathsf{E}[X_n^2 X_{n-1}^2] = n^2(n-1)^2/n^4 \to 1 \neq \mathsf{E}[X^2X^2] = 0$, Thus, by the Cauchy criterion, X_n^2 does not converge in m.s. sense.

   (c) Suppose $X \sim \text{Unif}[-1,1]$ and $X_n = X^n$. Does X_n converge almost surely as $n \to \infty$?

   Ans: Yes. $X_n(\omega) = X(\omega)^n \to 0$ for all ω except that for which $X(\omega) = 1$ or $X(\omega) = -1$, which belong to set of measure 0. Thus $X_n \xrightarrow{a.s.} 0$.

   (d) Suppose $X_n \xrightarrow{d.} X$, and a_n is a deterministic sequence such that $a_n \to a$ as $n \to \infty$. Does $X_n + a_n$ necessarily converge in distribution as $n \to \infty$?

   Ans: Yes. Using characteristic functions, we have $\mathsf{E}[e^{juX_n}] \to \mathsf{E}[e^{juX}]$ for all $u \in \mathbb{R}$. Thus

   $$\mathsf{E}[e^{j(X_n+a_n)u}] = e^{ja_nu}\mathsf{E}[e^{jX_nu}] \to e^{jau}\mathsf{E}[e^{jXu}] = \mathsf{E}[e^{j(X+a)u}]$$

   which means that $X_n + a_n \xrightarrow{d.} X + a$.

2. Let $X_1, X_2, \ldots$ be i.i.d. Bernoulli random variables, with

   $$\mathsf{P}\{X_n = 0\} = \frac{3}{4} \quad \text{and} \quad \mathsf{P}\{X_n = 1\} = \frac{1}{4}$$

   Suppose $S_n = \sum_{i=1}^n X_i$.

   (a) Find $M_X(\theta)$, the moment generating function of X_n.

   Ans: $M_X(\theta) = \mathsf{E}[e^{\theta X_n}] = \frac{1}{4}e^\theta + \frac{3}{4}$.

   (b) Use the Central Limit Theorem to find an approximation for $\mathsf{P}\{S_{100} \geq 50\}$ in terms of the $Q(\cdot)$ function.

   Ans: $\mu = \mathsf{E}[X_n] = \frac{1}{4}$ and $\sigma^2 = \text{Var}(X_n) = \mathsf{E}[X_n^2] - \mu^2 = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$. Thus, by the Central Limit Theorem, $(S_{100} - 100\mu)/(10\sigma)$ is approximately $\mathcal{N}(0,1)$. Therefore,

   $$\mathsf{P}\{S_{100} > 50\} = \mathsf{P}\left\{\frac{S_{100} - 100\mu}{10\sigma} > \frac{50 - n\mu}{10\sigma}\right\} \approx Q\left(\frac{50 - n\mu}{10\sigma}\right) = Q\left(\frac{10}{\sqrt{3}}\right)$$

(c) Now use the Chernoff Bound to show that

$$\mathsf{P}\{S_{100} \geq 50\} \leq \left(\frac{4}{3}\right)^{-50}$$

Ans: By the Chernoff Bound,

$$\mathsf{P}\{S_{100} \geq 50\} = \mathsf{P}\left\{\frac{S_{100}}{100} \geq \frac{1}{2}\right\} \leq e^{-100\,\ell(0.5)}$$

where $\ell(0.5)$ is obtained by maximizing

$$0.5\,\theta - \ln M_X(\theta) = 0.5\theta - \ln(3 + e^{\theta}) + \ln(4)$$

Taking the derivative and setting it equal to zero, we obtain that the optimizing θ^* satisfies

$$0.5 = \frac{e^{\theta^*}}{3 + e^{\theta^*}} \implies \theta^* = \ln 3$$

Thus $\ell(0.5) = 0.5 \ln 3 - \ln(3/2) = 0.5 \ln 4 - 0.5 \ln 3$, and the upper bound follows.

3. (12 pts) Suppose X, Y have joint pdf

$$f_{X,Y}(x,y) = \begin{cases} 6x & \text{if } x, y \geq 0 \text{ and } x + y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $\mathsf{E}[X|Y]$.
Ans: $f_{X,Y}(x,y) = 6x \; 1\!\!1_{\{0 \leq y \leq 1\}} \; 1\!\!1_{\{0 \leq x \leq 1-y\}}$. Thus

$$f_y(y) = \int_0^{1-y} 6x dx \; 1\!\!1_{\{0 \leq y \leq 1\}} = 3(1-y)^2 \; 1\!\!1_{\{0 \leq y \leq 1\}}$$

and for $0 \leq y \leq 1$,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_y(y)} = \frac{2x}{(1-y)^2} \; 1\!\!1_{\{0 \leq x \leq 1-y\}}.$$

Therefore, for $0 \leq y \leq 1$,

$$\mathsf{E}[X|Y = y] = \int_0^{1-y} x f_{X|Y}(x|y) dx = \frac{2}{3} \frac{(1-y)^3}{(1-y)^2} = \frac{2}{3}(1-y)$$

and $\mathsf{E}[X|Y] = \frac{2}{3}(1-Y)$.

(b) Find the MSE achieved by $\mathsf{E}[X|Y]$, i.e. find the minimum MSE.
Ans: It is easy to see that $f_x(x) = 6x(1-x) 1\!\!1_{\{0 \leq x \leq 1\}}$. Thus, the minimum MSE is given by

$$\mathsf{E}[X^2] - \mathsf{E}[(\mathsf{E}[X|Y])^2] = \int_0^1 6x^3(1-x)dx - \frac{4}{9}\int_0^1 3(1-y)^4 dy = \frac{3}{10} - \frac{4}{15} = \frac{1}{30}.$$

(c) Find $\hat{\mathsf{E}}[X|Y]$.
Ans: Since $\mathsf{E}[X|Y]$ is linear in Y, $\hat{\mathsf{E}}[X|Y] = \mathsf{E}[X|Y]$.

4. (14 pts) Suppose X, Y_1, Y_2 are *zero-mean* jointly Gaussian with covariance matrix

$$\text{Cov}\left(\begin{bmatrix} X \\ Y_1 \\ Y_2 \end{bmatrix}\right) = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

(a) Find $\mathsf{P}\{Y_1 + Y_2 - X \geq 10\}$ in terms of the $Q(\cdot)$ function.

Ans: Let $W = Y_1 + Y_2 - X$. Then W is Gaussian with $\mathsf{E}[W] = 0$ and

$$\text{Var}(W) = \mathsf{E}[W^2] = \mathsf{E}[Y_1^2] + \mathsf{E}[Y_1^2] + \mathsf{E}[X^2] + 2\mathsf{E}[Y_1 Y_2] - 2\mathsf{E}[XY_1] - 2\mathsf{E}[XY_2] = 1+1+4+0+2+2 = 10.$$

Thus $\mathsf{P}\{Y_1 + Y_2 - X \geq 10\} = Q(\sqrt{10})$.

(b) Find $\mathsf{E}[X|Y_1]$ and $\mathsf{E}[X|Y_2]$.

Ans: $\mathsf{E}[X|Y_1] = 0 + \text{Cov}(X, Y_1)\text{Cov}(Y_1)^{-1}(Y - 0) = -Y_1$. Similarly, $\mathsf{E}[X|Y_2] = -Y_2$.

(c) Find $f_{X|Y_1,Y_2}(x|y_1, y_2)$.

Ans: We know that given $Y_1 = y_1$, $Y_2 = y_2$, X is Gaussian with mean $E[X|Y_1 = y_1, Y_2 = y_2]$, and variance equal to $\text{Cov}(e)$, with $e = X - E[X|Y_1, Y_2]$. Now, with $\underline{Y} = [Y_1 Y_2]^\top$,

$$E[X|\underline{Y} = \underline{y}] = 0 + \text{Cov}(X, \underline{Y})\text{Cov}(\underline{Y})^{-1}[\underline{y}] = [-1\ -1]\underline{y} = -y_1 - y_2.$$

(Note: we could have concluded this from part (b) using linear innovations.) Similarly,

$$\text{Cov}(e) = \text{Cov}(X) - \text{Cov}(X, \underline{Y})\text{Cov}(\underline{Y})^{-1}\text{Cov}(\underline{Y}, X) = 4 - [-1\ -1]\ [-1\ -1]^\top = 2$$

Thus $f_{X|Y_1,Y_2}(x|y_1, y_2) \sim \mathcal{N}(-y_1 - y_2, 2)$.

(d) Find $\mathsf{P}(\{X \geq 2\}|\{Y_1 + Y_2 = 0\})$ in terms of the $Q(\cdot)$ function.

Ans: The straightforward way to do this problem is to define $V = Y_1 + Y_2$, note that X and V are jointly Gaussian, find the conditional distribution of X given V using the MMSE approach, and then compute the above probability. But based on the result of part (c), we can conclude that $f_{X|V}(x|v) \sim \mathcal{N}(-v, 2)$. Thus $\mathsf{P}(\{X \geq 2\}|\{Y_1 + Y_2 = 0\}) = \mathsf{P}(\{X \geq 2\}|\{V = 0\}) = Q(\sqrt{2})$.

(e) Let $Z = Y_1^2 + Y_2^2$. Find $\hat{\mathsf{E}}[X|Z]$.

Ans: Note that $\text{Cov}(X, Z) = \mathsf{E}[XY_1^2] + \mathsf{E}[XY_2^2] = 0$, since for $i = 1, 2$,

$$\mathsf{E}[XY_i^2] = \mathsf{E}[\mathsf{E}[XY_i^2|Y_i]] = \mathsf{E}[Y_i^2\mathsf{E}[X|Y_i]] = -\mathsf{E}[Y_i^3] = 0$$

Thus

$$\hat{\mathsf{E}}[X|Z] = \mathsf{E}[X] - 0 = 0$$