ECE 534: SPRING 2017
MIDTERM 1

PLEASE WRITE YOUR NAME AND STUDENT ID NUMBER HERE:

The exam duration is 2 hours. You are allowed to have one sheet of notes with you. Please write each step in your derivations on the papers attached. You will not get credit for solutions which do not contain sufficient details to judge the correctness of your derivations.

- **Problem 1 (10 points)** Consider an experiment in which a coin is tossed independently many times. Let $P(H_n)$ be the probability of obtaining head at the n-th toss, and similarly, let $P(T_n)$ be the probability of obtaining tail at the n-th toss. Show that if $P(H_n) = 1/n$, for $n \geqslant 1$, then infinitely many heads will be observed almost surely. How many heads do you expect to observe for $P(H_n) = 1/n^2$?

- **Problem 2 a) (15 points)** Let each X_n, $n = 1, 2, \ldots$, in a sequence of random variables follow a normal $\mathcal{N}(0, 1/n)$ distribution (Note that the variance of the normal distribution changes with the index n). Does the sequence converge in distribution? Please make sure to analyze the value of the CDFs at the point $x = 0$.

- **Problem 2 b) (15 points)** Let X be a standard normal random variable, and let $X_n = -X$ for all $n = 1, 2, \ldots$. Does the sequence $\{X_n\}$ converge in distribution to X? Does the sequence converge in probability to X? Please comment on the relationship between your two findings.

- **Problem 2 c) (15 points)** Let X_n, for $n = 1, 2, \ldots$, be random variables with the following distribution: $P\{X_n = n^2\} = 1/n$ and $P\{X_n = 0\} = 1 - 1/n$. Does X_n converge in probability? If it does converge to some random variable X_∞, is it true that $E[X_n] \to E[X_\infty]$?

- **Problem 3 (25 points)** Let $X_1, X_2, \ldots$ be independent RVs, each with the Cauchy distribution and density function

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \; x \in \mathbb{R}.$$

The characteristic function of a standard Cauchy variable is $\Phi(u) = \exp(-|u|)$. Let $S_n = \sum_{i=1}^{n} X_i$. Determine for which cases of sequences of random variables does convergence in distribution hold, and if it holds, specify the limiting distribution.
i) S_n/n; ii) S_n/n^2; iii) $S_n/\sqrt{n}$.
You may want to start by computing the Characteristic function of the RV S_n/n^θ, for constant θ.

- **Problem 4 (20 points)** Let X_n, $n = 1, 2, \ldots$, be a sequence of independent random variables with Bernoulli distribution $B(1/2)$. If $S_n = \sum_{i=1}^{n} X_i$, evaluate the Chebyshev and Chernoff bound for $P\{S_n/n \geqslant (1/2) + \epsilon\}$, where $\epsilon > 0$.

- **Problem 5 a) (25 points)** Consider an additive noise channel for which $Y = X + Z$, where the signal X is normal $\mathcal{N}(0, P)$ and the noise Z has zero mean and variance N. Assume furthermore that the variables X and Z are independent. Find a distribution of Z that maximizes the minimum MSE of estimating X given Y , i.e., the distribution of the worst-case noise Z that has the given mean and variance. (Hint: Think of Gaussian variables and linear estimators and make sure that you justify your answer!)

- **Problem 5 b) (25 points)** Assume that X in uniformly distributed in $[1, 2]$, i.e., $X \sim U[1, 2]$ and that given $X = x$, Y is distributed exponentially with mean value $1/(2x)$. Find the minimum mean square (MMSE) estimator of X given Y, as well as the linear MMSE of X given Y.