Solutions to Final Exam

1. (24 pts, equally weighted parts) True or False.

(a) If $U_1, U_2, \ldots,$ is a sequence i.i.d. Unif[0,1] random variables and $X_n = (U_n)^n$, $n \geq 1$, then $X_n$ converges in probability as $n \to \infty$.
   \textbf{Ans:} True. In fact $X_n \xrightarrow{m.s.} 0$, since $E[X_n^2] = E[U_n^2] = 1/(2n + 1) \to 0$ as $n \to \infty$.

(b) Suppose $E[X_n^2] < \infty$, for all $n$. If $X_n \xrightarrow{p.} c$, where $c$ is a deterministic constant, then $X_n \xrightarrow{m.s.} c$ as well.
   \textbf{Ans:} False. Consider $\Omega = [0,1]$ with the uniform probability measure, and let $X_n = n \mathbb{1}_{(\omega \in [0,1/n])}$. Then $X_n \xrightarrow{a.s.} 0$ and hence $X_n \xrightarrow{p.} 0$, but $E[X_n^2] = n \to \infty$ as $n \to \infty$.

(c) If $(X_t, t \in \mathbb{R})$ is Gaussian random process with covariance function $C_X(s,t) = st + \min\{s,t\}$, then $(X_t)$ cannot be a Markov process.
   \textbf{Ans:} False. A Gauss-Markov process needs to satisfy, for $r < s < t$
   \[ C_X(r,t) = \frac{C_X(r,s) C_X(s,t)}{C_X(s,s)} \]
   It is easy to check that the given covariance function does satisfy the condition and is indeed Markov.

(d) If $X$ and $Y$ are jointly Gaussian random variables with finite second moments, then
   \[ E[(X - E[X|Y])^2] = E[(X - \hat{X}[X|Y,Y^2])^2] \]
   \textbf{Ans:} True. Since $X$ and $Y$ are jointly Gaussian, the MMSE estimate is linear. So adding a quadratic term to the LMMSE estimator cannot decrease the MSE below that obtained by just having the linear term.

(e) The function $R(\tau) = |\sin(\tau)|$ is a valid auto-correlation function for a WSS process.
   \textbf{Ans:} False. $R(0) = 0 < R(\pi/2) = 1$.

(f) The function $S(\omega) = e^{-|\omega|}|\sin(\omega)|$ is a valid power spectral density for a WSS process.
   \textbf{Ans:} True. Since $S(\omega)$ is real-valued and $\geq 0$ for all $\omega$.

(g) A time-homogenous discrete-state Markov process $(X_t)$ satisfies $\pi(t) = \pi$ for some distribution $\pi$. Then $(X_t)$ must be a (strictly) stationary process.
   \textbf{Ans:} True. For any $n$ and $t_1 < t_2 < \cdots < t_n$, the joint distribution of $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ depends on the marginal of $X_{t_i}$ and the transition matrices $H(t_1, t_2), H(t_2, t_3), \ldots, H(t_{n-1}, t_n)$, all of which are invariant if we replace $t_i$ by $t_i + \tau$, $i = 1,2,\ldots,n$.

(h) For zero-mean jointly WSS $(X_t)$ and $(Y_t)$, the noncausal Wiener filter for optimum linear estimation of $X_t$ given $\{Y_s : s \in \mathbb{R}\}$ is necessarily time-invariant.
   \textbf{Ans:} True. It is easy to see that the linear Kernel $h(u, v)$ for optimum estimation of $X_t$ given $\{Y_s : s \in \mathbb{R}\}$ must be the same that for estimation of $X_{t+\tau}$ from $\{Y_s : s \in \mathbb{R}\} = \{Y_{s+\tau} : s \in \mathbb{R}\}$, which means that $h(u, v) = h(u + \tau, v + \tau)$ for all $\tau \in \mathbb{R}$.
2. (12 pts) CLT and Chernoff Bound. Let \( \{X_k : k \geq 0\} \) be a sequence of i.i.d. random variables with
\[
P\{X_k = -1\} = \frac{1}{4} \quad P\{X_k = 0\} = \frac{1}{2} \quad P\{X_k = 1\} = \frac{1}{4}
\]
Suppose \( S_n = \sum_{k=1}^{n} X_k \).

(a) Find \( M_X(\theta) \), the moment generating function of \( X_k \).
\[
\text{Ans: } M_X(\theta) = E[e^{\theta X_n}] = \frac{1}{4} (e^{\theta} + e^{-\theta}) + \frac{1}{2}.
\]
(b) Use the Central Limit Theorem to find an approximation for \( P\{S_{100} \geq 50\} \) in terms of the \( Q(\cdot) \) function.
\[
\text{Ans: } \mu = E[X_n] = 0 \text{ and } \sigma^2 = \text{Var}(X_n) = E[X_n^2] = \frac{1}{2}.
\]
Thus, by the Central Limit Theorem, \( (S_{100}/10\sigma) \) is approximately \( \mathcal{N}(0, 1) \). Therefore,
\[
P\{S_{100} > 50\} = P\left\{ \frac{S_{100}}{10\sigma} > \frac{50}{10\sigma}\right\} \approx Q\left(\frac{5\sqrt{2}}{3}\right)
\]
(c) Now use the Chernoff Bound to find a bound on \( P\{S_{100} \geq 50\} \).
\[
\text{Ans: } \text{By the Chernoff Bound,}
\]
\[
P\{S_{100} \geq 50\} = P\left\{ \frac{S_{100}}{100} \geq \frac{1}{2}\right\} \leq e^{-100 \ell(0.5)}
\]
where \( \ell(0.5) \) is obtained by maximizing
\[
0.5 \theta - \ln M_X(\theta) = 0.5\theta - \ln(2 + e^{\theta} + e^{-\theta}) + \ln(4)
\]
Taking the derivative and setting it equal to zero, we obtain that the optimizing \( \theta^* \) satisfies
\[
0.5 = \frac{e^{\theta^*} - e^{-\theta^*}}{2 + e^{\theta^*} + e^{-\theta^*}}
\]
Setting \( x = e^{\theta^*} \) reduces the above equation to the quadratic \( x^2 - 2x - 3 = 0 \), which has the solutions \( x = 3 \) and \( x = -1 \). Since \( x \) has to be positive, we conclude that \( x = 3 \implies \theta^* = \ln 3 \). Thus \( \ell(0.5) = 0.5 \ln 3 - \ln(4/3) = \ln(3\sqrt{3}/4) \). Therefore,
\[
P\{S_{100} \geq 50\} \leq \left(\frac{3\sqrt{3}}{4}\right)^{-100}
\]

3. (14 pts) Linear Innovations. Let \( \{Y_k : k \geq 1\} \) be a discrete-time zero-mean WSS random process with ACF
\[
R_Y(k) = (0.5)^{|k|}
\]
(a) Find the linear innovations sequence \( \tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3 \) corresponding to the first three samples of the process \( Y_1, Y_2, Y_3 \).
\[
\text{Ans: } \tilde{Y}_1 = Y_1, \text{ and } \tilde{Y}_2 = Y_2 - \mathbb{E}[Y_2 | \tilde{Y}_1] = Y_2 - \mathbb{E}[Y_2 | Y_1]. \text{ Now, } \text{Var}(Y_1) = 1 \text{ and } \text{Cov}(Y_2, Y_1) = 0.5. \text{ Thus } \mathbb{E}[Y_2 | Y_1] = 0.5Y_1, \text{ and } \tilde{Y}_2 = Y_2 - 0.5Y_1. \text{ Now by linear innovations applied recursively,}
\]
\( \tilde{Y}_3 = Y_3 - (\mathbb{E}[Y_3 | \tilde{Y}_2] + \mathbb{E}[Y_3 | \tilde{Y}_1]) \). Since \( \text{Cov}(Y_3, Y_1) = 0.25, \mathbb{E}[Y_3 | Y_1] = \mathbb{E}[Y_3 | Y_2] = 0.25Y_1 \). Furthermore, \( \text{Var}(Y_2) = \text{Var}(Y_2) + 0.25 \text{Var}(Y_1) = \frac{3}{4} \), and \( \text{Cov}(Y_3, Y_2) = \mathbb{E}[Y_3 Y_2] - 0.5E[Y_2 Y_1] = 0.5 - 0.125 = \frac{3}{8} \), which means that \( \mathbb{E}[Y_3 | \tilde{Y}_2] = \frac{3}{8} \tilde{Y}_2 \). Thus \( \tilde{Y}_3 = Y_3 - 0.25Y_1 - 0.5(Y_2 - 0.5Y_1) = Y_3 - 0.5Y_2. \)
(b) Now suppose \( X \) is a zero mean random variable with finite second moment satisfying
\[
E[XY_1] = 1, \quad E[XY_2] = 0.5, \quad E[XY_3] = 0.25
\]
Find the LMMSE estimate \( \hat{E}[X|Y_1, Y_2, Y_3] \).
\[\text{Ans:} \quad \hat{E}[X|Y_1, Y_2, Y_3] = \hat{E}[X|\hat{Y}_1] + \hat{E}[X|\hat{Y}_2] + \hat{E}[X|\hat{Y}_3]. \]
Now, \( \hat{E}[X|\hat{Y}_1] = \hat{E}[X|Y_1]E[XY_1]\)Var(\( Y_1 \))\(^{-1}\)\( Y_1 = Y_1. \) Furthermore, it is easy to see that \( \hat{E}[X|Y_2] = \hat{E}[X|\hat{Y}_2] = 0 \), which means that \( \hat{E}[X|\hat{Y}_3] = 0 \). Thus \( \hat{E}[X|Y_1, Y_2, Y_3] = Y_1. \)

4. (16 pts) Poisson process. Let \( (N_t : t \geq 0) \) be a Poisson process with parameter \( \lambda = 1. \)

(a) Find \( P\{N_3 \leq 2 \mid N_1 \geq 1\}. \)
\[\text{Ans:} \quad P\{N_3 \leq 2 \mid N_1 \geq 1\} = \frac{P\{N_3 \leq 2, N_1 \geq 1\}}{P\{N_1 \geq 1\}}\]
Now, \( P\{N_1 \geq 1\} = 1 - P\{N_1 = 0\} = 1 - e^{-1}, \) and using the independent increment property of \( (N_t), \)
\[P\{N_3 \leq 2, N_1 \geq 1\} = P\{N_1 = 2\}, \quad N_3 - N_1 = 0\} + P\{N_1 = 1, \quad N_3 - N_1 \leq 1\} = \cdots = \frac{7}{2} e^{-3}\]
Thus \( P\{N_3 \leq 2 \mid N_1 \geq 1\} = \frac{7}{2} e^{-3}. \)

(b) Find \( P\{N_1 \geq 1 \mid N_3 \leq 2\}. \)
\[\textbf{Ans:} \quad P\{N_1 \leq 2\} = e^{-3} + 3e^{-3} + \frac{9}{2} e^{-3} = \frac{17}{2} e^{-3}. \)
Thus \( P\{N_1 \geq 1 \mid N_3 \leq 2\} = \frac{7}{17}. \)

(c) Now suppose we define the random variable \( Z \) via the m.s. integral
\[Z = \int_{0}^{1} N_t dt\]
Find the LMMSE estimate \( \hat{E}[N_2|Z]. \)
\[\text{Ans:} \quad \text{The autocovariance function of } (N_t) \text{ is given by } C_N(s, t) = \min(s, t). \]
\[E[Z] = \int_{t=0}^{1} t dt = \frac{1}{2}, \quad \text{Var}(Z) = \int_{0}^{1} \int_{0}^{1} C_N(s, t) dt ds = \int_{0}^{1} \int_{0}^{1} \min(s, t) dt ds = \frac{1}{3}\]
Furthermore,
\[\text{Cov}(N_2, Z) = \int_{t=0}^{1} C_N(t, 2) dt = \int_{t=0}^{1} t dt = \frac{1}{2}\]
Thus \( \hat{E}[N_2|Z] = 2 + \frac{1}{4} 3 (Z - \frac{1}{2}) = \frac{3}{4} Z + \frac{5}{4}. \)

5. (20 pts) FSMP. Consider a time-homogeneous discrete-time Markov process \( (X_k : k \geq 0) \) with state space \( S = \{-1, 0, 1\} \) and one-step probability transition matrix \( P \) given by
\[
P = \begin{bmatrix}
0.2 & 0.8 & 0 \\
0.4 & 0.2 & 0.4 \\
0 & 0.8 & 0.2
\end{bmatrix}
\]

(a) Find the equilibrium distribution \( \pi. \)
\[\textbf{Ans:} \quad \text{Using the fact that } \pi = \pi P \text{ and } \pi e = 1, \text{ it is easy to show that } \pi_{-1} = \pi_1 = \frac{1}{4} \text{ and } \pi_0 = \frac{1}{2}. \]
For the remaining parts, assume that \( X_0 \) has the equilibrium distribution.

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(b) Determine whether or not \((X_k)\) is a martingale.

**Ans:** No. For example, \(E[X_2|X_1 = -1] = (0.2)(-1) + (0.8)(0) = -0.2 \neq -1\).

(c) Find the joint distribution of \(X_1\) and \(X_2\). (You may want to put the values in a table.)

**Ans:** \(P\{X_2 = j, X_1 = i\} = \pi_{i,j}\). Thus the joint pmf is described by table

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(d) Let the discrete-time process \((Y_k : k \geq 0)\) be defined by

\[Y_k = X_1 + kX_2, \quad k \geq 0\]

Find the mean and autocorrelation function of \((Y_k)\).

**Ans:** \(E[X_1] = E[X_2] = 0, E[X_1^2] = E[X_2^2] = \frac{1}{2}\), and \(E[X_1X_2] = (-1)(-1)(0.05) + (1)(1)(0.05) = 0.1\). Thus

\[E[Y_k] = 0, \quad R_Y(k, m) = E[Y_kY_m] = \frac{1}{2} + \frac{km}{2} + (0.1)(k + m)\]

(e) Find \(E[Y_2|Y_1, Y_0]\).

**Ans:** \(Y_2 = X_1 + 2X_2 = 2Y_1 - Y_0\). Thus \(E[Y_2|Y_1, Y_0] = 2Y_1 - Y_0\).

(f) Determine whether or not \((Y_k)\) is a Markov process.

**Ans:** No, since \(E[Y_2|Y_1, Y_0]\) depends on both \(Y_1\) and \(Y_0\). In particular

\[E[Y_2|Y_1 = 1, Y_0 = 1] = 1 \neq E[Y_2|Y_1 = 1] = 1 + E[X_2|Y_1 = 1] = 1 + \frac{1}{2}\]

6. (14 pts) **Filtering.** Consider a zero-mean WSS process \((X_t)\) with autocorrelation function

\[R_X(\tau) = \frac{1}{2}e^{-|\tau|}\]

Suppose \((X_t)\) is passed through a linear time-invariant system with transfer function

\[H(\omega) = \frac{1}{3 + j\omega}\]

to produce the output process \((Y_t)\).

(a) Find \(S_{YX}(\omega)\) and use it to find \(R_{YX}(\tau)\).

**Ans:**

\[S_{YX}(\omega) = H(\omega)S_X(\omega) = \frac{1}{3 + j\omega} \frac{1}{1 + \omega^2} = \frac{1}{4} \frac{1}{1 + \omega} + \frac{1}{8} \frac{1}{1 - j\omega} - \frac{1}{8} \frac{1}{3 + j\omega}\]

where the last equality follows from using partial fractions. Applying the inverse Fourier transform

\[R_{YX}(\tau) = \left(\frac{1}{4}e^{-\tau} - \frac{1}{8}e^{-3\tau}\right) \mathbb{1}_{\{\tau \geq 0\}} + \frac{1}{8}e^\tau \mathbb{1}_{\{\tau < 0\}}\]
(b) Find $S_Y(\omega)$ and use it to find $R_Y(\tau)$.

**Ans:** $S_Y(\omega) = S_X(\omega)|H(\omega)|^2$. Using the Fourier transform pairs given to you

$$S_Y(\omega) = \frac{1}{9 + \omega^2} \frac{1}{1 + \omega^2} = \frac{1}{8} \left[ \frac{1}{1 + \omega^2} - \frac{1}{9 + \omega^2} \right] = \frac{1}{8} \left[ \frac{1}{2} \frac{2}{1 + \omega^2} - \frac{1}{6} \frac{6}{9 + \omega^2} \right]$$

and therefore

$$R_Y(\tau) = \frac{1}{16} e^{-|\tau|} - \frac{1}{48} e^{-3|\tau|}$$

(c) Find the LMMSE estimate $\hat{E}[X_2|Y_1]$.

**Ans:** $E[X_2Y_1] = E[Y_1X_2] = R_{YX}(-1) = \frac{1}{6} e^{-1}$ and $\text{Var}(Y_1) = R_Y(0) = \frac{1}{24}$. Thus

$$\hat{E}[X_2|Y_1] = 0 + \frac{1}{8} e^{-1} 24(Y_1 - 0) = 3e^{-1}Y_1$$

7. (Extra credit – attempt only if you have time; I will not grade your answer if you have not finished the rest of the exam)

*The Cliff-Hanger.* A drunken man is near a cliff. From where he stands, one step toward the cliff would send him over the edge. He takes a random step either towards or away from the cliff. At any step, his probability of taking a step away from the cliff is $p$, and of a step towards the cliff is $(1 - p)$. Find the probability that he will escape unharmed as a function of $p$, for the entire range $0 \leq p \leq 1$.

**Ans:** This is essentially the Gambler’s ruin problem with initial wealth of $k = 1$ and goal of $b = \infty$. It is easier to calculate the probability that the man will fall off the cliff, which we denote by $\rho$. Using the formula we derived in class, we get (for $p \neq \frac{1}{2}$)

$$\rho = \lim_{b \to \infty} \left( \frac{1-p}{p} \right)^b \frac{\left( \frac{1-p}{p} \right)^b}{1 - \left( \frac{1-p}{p} \right)^b}$$

If $0 \leq p < \frac{1}{2}$, $\left( \frac{1-p}{p} \right)^b$ converges to $\infty$ as $b \to \infty$, which means that $\rho = 1$.

If $\frac{1}{2} < p \leq 1$, $\left( \frac{1-p}{p} \right)^b$ converges to 0 as $b \to \infty$, which means that $\rho = \frac{1-p}{p}$.

For $p = \frac{1}{2}$, we use the boundary conditions to get $\rho = \lim_{b \to \infty} 1 - \frac{1}{b} = 1$.

We can also solve the problem directly without using the Gambler’s ruin solution. Note that the probability of falling off the cliff starting two steps away is simply $\rho^2$. Thus $\rho = (1 - p) + \rho^2 p$, which we can solve to get $\rho = 1$ or $\rho = (1 - p)/p$. If $p < \frac{1}{2}$, the second solution is impossible since $\rho$ has to be $\leq 1$. For $p = 1$, it is clear that $\rho = 0$. Now, we can argue that $\rho$ should be continuous in $p$ to conclude that for $p \geq \frac{1}{2}$, $\rho = (1 - p)/p$. 

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