ECE 534 **Fall 2009**

December 16, 2009

Final Exam

- You have 3 hours to complete this exam.
- Don't forget to put your name on the answer booklet.
- You are allowed 3 sheets of notes (8.5"×11", both sides).
- Calculators laptop computers, PDA's, etc. are not permitted.
- Maximum possible score is 100.
- Neatness counts, especially for partial credit towards incorrect solutions.
- You may find the following Fourier transform pairs to be useful:

$$\text{For } a > 0, \quad e^{-at}\mathbb{1}_{\{t\geq 0\}} \leftrightarrow \frac{1}{a+j\omega}, \qquad e^{at}\mathbb{1}_{\{t<0\}} \leftrightarrow \frac{1}{a-j\omega} \quad \text{and} \qquad e^{-a|t|} \leftrightarrow \frac{2a}{a^2+\omega^2}$$

1. (24 pts, equally weighted parts) *True or False.* Determine if the following statements are True or False. You need to justify your answer clearly to get credit – provide a short proof if you say the statement is True, and a counter-example if you say the statement is False. Just stating "True" or "False" without any justification will get zero credit.

   (a) If $U_1, U_2, \ldots,$ is a sequence i.i.d. Unif[0,1] random variables and $X_n = (U_n)^n$, $n \geq 1$, then X_n converges in probability as $n \to \infty$.

   (b) Suppose $\mathsf{E}[X_n^2] < \infty$, for all n. If $X_n \xrightarrow{p.} c$, where c is a deterministic constant, then $X_n \xrightarrow{m.s.} c$ as well.

   (c) If $(X_t, t \in \mathbb{R})$ is Gaussian random process with covariance function $C_X(s,t) = st + \min\{s,t\}$, then (X_t) *cannot* be a Markov process.

   (d) If X and Y are jointly Gaussian random variables with finite second moments, then

   $$\mathsf{E}[(X - \mathsf{E}[X|Y])^2] = \mathsf{E}[(X - \hat{\mathsf{E}}[X|Y, Y^2])^2]$$

   (e) The function $R(\tau) = |\sin(\tau)|$ is a valid auto-correlation function for a WSS process.

   (f) The function $S(\omega) = e^{-|\omega|}|\sin(\omega)|$ is a valid power spectral density for a WSS process.

   (g) A time-homogenous discrete-state Markov process (X_t) satisfies $\underline{\pi}(t) = \underline{\pi}$ for some distribution $\underline{\pi}$. Then (X_t) must be a (strictly) stationary process.

   (h) For zero-mean jointly WSS (X_t) and (Y_t), the noncausal Wiener filter for optimum linear estimation of X_t given $\{Y_s : s \in \mathbb{R}\}$ is necessarily *time-invariant*.

2. (12 pts) *CLT and Chernoff Bound.* Let $\{X_k : k \geq 0\}$ be a sequence of i.i.d. random variables with

$$\mathsf{P}\{X_k = -1\} = \frac{1}{4} \quad \mathsf{P}\{X_k = 0\} = \frac{1}{2} \quad \mathsf{P}\{X_k = 1\} = \frac{1}{4}$$

Suppose $S_n = \sum_{k=1}^{n} X_k$.

(a) Find $M_X(\theta)$, the moment generating function of X_k.

(b) Use the Central Limit Theorem to find an approximation for $\mathsf{P}\{S_{100} \geq 50\}$ in terms of the $Q(\cdot)$ function.

(c) Now use the Chernoff Bound to find a bound on $\mathsf{P}\{S_{100} \geq 50\}$.

3. (14 pts) *Linear Innovations.* Let $(Y_k : k \geq 1)$ be a discrete-time *zero-mean* WSS random process with ACF

$$R_Y(k) = (0.5)^{|k|}$$

(a) Find the linear innovations sequence $\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3$ corresponding to the first three samples of the process Y_1, Y_2, Y_3.

(b) Now suppose X is a *zero mean* random variable with finite second moment satisfying

$$\mathsf{E}[XY_1] = 1, \quad \mathsf{E}[XY_2] = 0.5, \quad \mathsf{E}[XY_3] = 0.25$$

Find the LMMSE estimate $\hat{\mathsf{E}}[X|Y_1, Y_2, Y_3]$.

4. (16 pts) *Poisson process.* Let $(N_t : t \geq 0)$ be a Poisson process with parameter $\lambda = 1$.

(a) Find $\mathsf{P}\{N_3 \leq 2 \mid N_1 \geq 1\}$.

(b) Find $\mathsf{P}\{N_1 \geq 1 \mid N_3 \leq 2\}$.

(c) Now suppose we define the random variable Z via the m.s. integral

$$Z = \int_0^1 N_t dt$$

Find the LMMSE estimate $\hat{\mathsf{E}}[N_2|Z]$.

5. (20 pts) *FSMP.* Consider a time-homogeneous discrete-time Markov process $(X_k : k \geq 0)$ with state space $\mathcal{S} = \{-1, 0, 1\}$ and one-step probability transition matrix P given by

$$P = \begin{bmatrix} 0.2 & 0.8 & 0 \\ 0.4 & 0.2 & 0.4 \\ 0 & 0.8 & 0.2 \end{bmatrix}$$

(a) Find the equilibrium distribution $\underline{\pi}$.

For the remaining parts, assume that X_0 has the equilibrium distribution.

(b) Determine whether or not (X_k) is a martingale.

(c) Find the joint distribution of X_1 and X_2. (You may want to put the values in a table.)

(d) Let the discrete-time process $(Y_k : k \geq 0)$ be defined by

$$Y_k = X_1 + kX_2, \quad k \geq 0$$

Find the mean and autocorrelation function of (Y_k).

(e) Find $\mathsf{E}[Y_2|Y_1, Y_0]$.

(f) Determine whether or not (Y_k) is a Markov process.

6. (14 pts) *Filtering.* Consider a zero-mean WSS process (X_t) with autocorrelation function

$$R_X(\tau) = \frac{1}{2}e^{-|\tau|}$$

Suppose (X_t) is passed through a linear time-invariant system with transfer function

$$H(\omega) = \frac{1}{3 + j\omega}$$

to produce the output process (Y_t).

(a) Find $S_{YX}(\omega)$ and use it to find $R_{YX}(\tau)$.

(b) Find $S_Y(\omega)$ and use it to find $R_Y(\tau)$.

(c) Find the LMMSE estimate $\hat{\mathsf{E}}[X_2|Y_1]$.

7. (Extra credit – attempt only if you have time; I will not grade your answer if you have not finished the rest of the exam)

The Cliff-Hanger. A drunken man is near a cliff. From where he stands, one step toward the cliff would send him over the edge. He takes a random step either towards or away from the cliff. At any step, his probability of taking a step away from the cliff is p, and of a step towards the cliff is $(1-p)$. Find the probability that he will escape unharmed as a function of p, for the entire range $0 \leq p \leq 1$.