Midterm Exam I

ECE534 | Spring 2006
There are a total of five problems | Mar. 8, 7:00-8:30 pm
You are allowed one sheet (two pages) of notes; no calculators. | Each problem is worth 20 points

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Please put your NAME here:

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1. (a) Let X be a uniformly distributed random variable on $[0, 1]$. Find the characteristic function of X.

(b) Let Y_1, $Y_2, \ldots$ be a sequence of independent random variables uniformly distributed over $\{0, 1, 2, \ldots, 9\}$. Find the characteristic function of Y_i.

(c) Let $X_n = \sum_{i=1}^{n} Y_i 10^{-i}$. Find the characteristic function of X_n.

(d) Does the sequence X_1, X_2, $X_3, \ldots$ converge in distribution? If so, what is the limiting distribution? Clearly justify your answer.

(e) Does the sequence X_1, X_2, $X_3, \ldots$ converge almost surely. If so, what is the distribution of the random variable to which it converges? Clearly justify your answer. (Hint: The following fact may be useful: a sequence of non-decreasing, upper-bounded real numbers has a finite limit.)

2. Suppose you toss a fair toss coin n times. Let H_n be the number of heads seen in the n tosses and let T_n be the number of tails seen in the n tosses. Using the Chernoff bound, show that the probability $P(H_n - T_n > nx)$ is upper-bounded by

$$\left(\frac{1}{\sqrt{(1+x)^{1+x}(1-x)^{1-x}}}\right)^n.$$

Assume $0 < x < 1$. (Hint: Write $H_n - T_n$ as a sum of n i.i.d. random variables.)

3. Let X be a random variable with mean 0 and variance 1. We wish to estimate X given n observations. The observations are of the form

$$Y_i = X + W_i,$$

where W_i are independent random variables with mean 0 and variance σ^2, which are also independent of X.

(a) Show that the linear MMSE estimate of X is given by

$$\hat{X} = \frac{\sum_{i=1}^{n} Y_i}{n + \sigma^2}.$$

(b) Show that the error covariance is given by

$$\frac{\sigma^2}{n + \sigma^2}.$$

(c) For what joint distribution of X, W_1, $W_2, \ldots,$ is the estimate in part (a) the best MMSE estimate (not just the best linear MMSE estimate)? There may be more than one answer to this question, but you have to provide just one answer.

4. Suppose that X and W are two identically distributed random variables with finite second moments, and let $Y = X + W$. Find the linear MMSE estimate of X given Y. Does your answer depend on whether X and W are independent or not? (Hint: $\hat{E}(Y|Y) = \hat{E}(X|Y) + \hat{E}(W|Y)$. No major computations are necessary to solve this problem.)

5. Suppose that X and Y are uniformly distributed in the region given by

$$\{(x, y) : |x| \leq 0.5, \quad x^2 \leq y \leq x^2 + 1\}.$$

(a) Find the MMSE estimate of Y given X.

(b) Suppose we wish to estimate Y using a quadratic MMSE estimator, i.e., one of the form $aX^2 + bX + c$. Find the optimal values of a, b and c.

(Hint: No major computations are necessary to solve this problem.)