University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 434: Random Processes

Spring 2004

Midsemester Exam 2

Wednesday, April 21, 5:00–7:00pm, 165 Everitt Laboratory

READ THESE COMMENTS BEFORE STARTING THE EXAM!

- This is a **closed-book** exam! You are allowed two sheets of *handwritten* notes (both sides). Calculators should not be necessary, but feel free to use one.
- **Write your name on the answer booklet.**
- There are **five unequally weighted** problems for a total of **50 points**. A **bonus** problem worth **5 points** is also included. Problems are *not* necessarily in order of difficulty.
- A correct answer does not guarantee credit; an incorrect answer does not guarantee loss of credit. **Provide clear explanations, show all relevant work and justify your answers!** If we cannot make sense of your writing or reasoning, you may loose points.
- Read each problem carefully and *think* before performing detailed calculations.
- Only the supplied answer booklet is to be handed in. **No additional pages will be considered in the grading.** You may want to work things through in the blank areas of the exam and then neatly transfer to the answer sheet the work you would like us to look at.

Problem 1 (14/50, equally weighted parts)

This problem has **seven independent** true/false questions.

(a) A zero mean wide-sense stationary random process X_t with autocorrelation function $R_X(\tau) = e^{-|\tau|}$ is mean square differentiable.

False $\quad e^{-|\tau|}$ is not differentiable at $\tau = 0$.

(b) If X_t is a wide-sense stationary random process, then $Y_t = X_t + X_{t-4}$ is also wide-sense stationary.

True

$$E[Y_t Y_s] = E[(X_t + X_{t-4})(X_s + X_{s-4})]$$
$$= 2R_X(t-s) + R_X(t-s+4) + R_X(t-s-4)$$

(c) Consider a discrete-time random process X_k defined so that the X_k's are i.i.d. random variables with $P[X_k = 1] = P[X_k = -1] = \frac{1}{2}$. The continuous-time random process $Y(t) = X(\lfloor t \rfloor)$, where $\lfloor t \rfloor$ denotes the largest integer smaller or equal to t, is wide-sense stationary.

False

$$E[Y_{0.6} \cdot Y_{1.1}] = E[X_0] \cdot E[X_1] = 0$$
$$E[Y_1 \cdot Y_{1.5}] = E[X_1^2] = 1 \quad \ast$$

(d) The function $R_X(t, s) = \cos(t + s)$ *cannot* be the autocorrelation function of a random process X_t.

True $\quad R_X(t,t) = E[X_t^2] \geq 0 \quad$ But $R_X(t,t) = \cos 2t$.

(e) A wide-sense stationary random process with autocorrelation function $R_X(\tau) = \frac{1}{1+\tau^2}$ is m.s. differentiable.

True $\quad R_X''(\tau)$ exists and is continuous.

(f) A wide-sense stationary random process with autocorrelation function $R_X(\tau) = \frac{1}{1+\tau^2}$ is *not* mean ergodic.

False $\quad \lim_{|\tau| \to \infty} \frac{1}{1+\tau^2} = 0 \Rightarrow X(t)$ is mean ergodic

(g) The function

$$R_X(\tau) = \begin{cases} |\sin \pi\tau|, & |\tau| < 1, \\ 0, & \text{else} \end{cases}$$

is a valid autocorrelation function for a wide-sense stationary random process X_t.

False $\quad \because R_X(0) \geq |R_X(\tau)| \ (\forall \tau)$ is not satisfied.

Problem 2 (12/50, equally weighted parts)

Consider a discrete state discrete-time Markov chain X_k with state set $\{1,2,3\}$ and one-step transition probability matrix P given by

$$P = \begin{bmatrix} 1/8 & 7/8 & 0 \\ 1/2 & 1/4 & 1/4 \\ 0 & 7/8 & 1/8 \end{bmatrix}.$$

(Recall that $P(i,j)$, the ith-row, jth-column entry of P, denotes the probability of transitioning from state i to state j.)

(a) What is the (first order) stationary distribution π of the Markov chain?

(b) Assuming that the Markov chain is initially in the stationary distribution that you calculated in part (a) (i.e., assuming that $P[X_0 = i] = \pi(i)$ for $i = 1, 2, 3$), calculate the mean and autocorrelation function of the random process

$$Y_t = t \cdot X_0 + t^2 \cdot X_1 .$$

(c) Let $Z_t = Y_t - \mu_Y(t)$ be a zero mean random process. Is Z_t wide-sense stationary? Justify your answer.

(a) Let $\pi = (\pi_1, \pi_2, \pi_3)^T$

$\therefore \pi^T \cdot P = \pi^T \Rightarrow (P^T - I)\pi = 0$ Meanwhile $\pi_1 + \pi_2 + \pi_3 = 1$

$\therefore \pi = [\frac{4}{13}, \frac{7}{13}, \frac{2}{13}]^T$

(b) $E[Y_t] = t \cdot E[X_0] + t^2 \cdot E[X_1]$ $\left(\because E[X_0] = E[X_1] = \frac{4}{13} + \frac{7 \cdot 2}{13} + \frac{2 \cdot 3}{13} = \frac{24}{13}\right)$

$= \frac{24}{13}(t + t^2)$

$R_Y(t,s) = E[(tX_0 + t^2X_1)(sX_0 + s^2X_1)]$

$= t \cdot s\, E[X_0^2] + t^2 s^2 E[X_1^2] + (ts^2 + t^2 s)\, E[X_0 X_1]$

$E[X_0^2] = \frac{4}{13} \cdot 1 + \frac{7}{13} \cdot 4 + \frac{2}{13} \cdot 9 = \frac{50}{13}$

X_0	1	1	1	2	2	2	3	3	3
X_1	1	2	3	1	2	3	1	2	3
$P(X_0, X_1)$	$(\frac{4}{13} \cdot \frac{1}{8})$	$(\frac{4}{13} \cdot \frac{7}{8})$	0	$(\frac{7}{13} \cdot \frac{1}{2})$	3 $(\frac{7}{13} \cdot \frac{1}{4})$	$(\frac{7}{13} \cdot \frac{1}{4})$	0	$(\frac{2}{13} \cdot \frac{7}{8})$	$(\frac{2}{13} \cdot \frac{1}{8})$

$\therefore E[X_0 X_1] = \frac{4}{13} \cdot \frac{1}{8} + 2 \frac{4}{13} \frac{7}{8} + 2 \frac{7}{13} \cdot \frac{1}{2} + 4 \cdot \frac{7}{13} \frac{1}{4} + 6 \cdot \frac{7}{13} \frac{1}{4} + 6 \frac{2}{13} \frac{7}{8} + 9 \cdot \frac{2}{13} \cdot \frac{1}{8} = \frac{179}{52}$

$\therefore R_Y(t,s) = \frac{50}{13} ts(1+ts) + ts(s+t)\frac{179}{52}$

(c) $E[Z_t] \equiv 0$

$$R_Z(t,s) = R_Y(t,s) - \mu_Y(t)\cdot\mu_Y(s)$$

Note: when $s=0$, $R_Z(t,0) \equiv 0$ $(\forall t)$

But we can always find some $t>0$, $s>0$, $t \neq s$ such that

$R_Z(t,s) \neq 0$

$\therefore Z(t)$ is not WSS.

Problem 3 (12/50, equally weighted parts)

Let N_t, $0 \le t < +\infty$, be a Poisson random process with $N_0 = 0$ and rate $\lambda > 0$.

(a) Find the joint probability that $P(N_{t_1} = n_1, N_{t_2} = n_2)$ for $t_2 > t_1$.

(b) Let $X_t = \int_0^t N_s ds$. Sketch a typical sample path of X_t and find $E[X_t]$.

(c) Is X_t m.s. differentiable?

(d) Assume that λ is a random variable that takes values 1, 2, and 3 with equal probability. Find the mean and variance of N_t.

(a) For $n_2 \geq n_1 \geq 0$

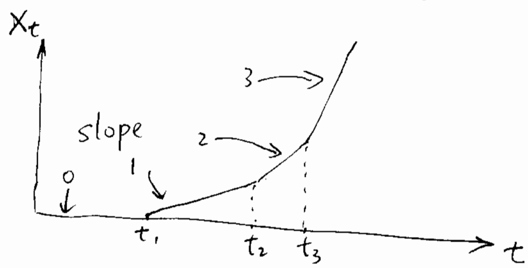
$$P(N_{t_1} = n_1, N_{t_2} = n_2) = P(N_{t_1} = n_1)\ P(N_{t_2} - N_{t_1} = n_2 - n_1)$$

$$= \frac{e^{-\lambda t_1} (\lambda t_1)^{n_1}}{n_1!} \frac{e^{-\lambda(t_2 - t_1)} (\lambda(t_2 - t_1))^{n_2 - n_1}}{(n_2 - n_1)!}$$

$$= \frac{e^{-\lambda t_2} \lambda^{n_2} t_1^{n_1} (t_2 - t_1)^{n_2 - n_1}}{n_1!\ (n_2 - n_1)!}$$

$P(N_{t_1} = n_1, N_{t_2} = n_2) = 0$ for $n_2 < n_1$

(b)



$$E[X_t] = \int_0^t \lambda s\, ds = \frac{\lambda t^2}{2}$$

(c) Yes. $\because$ N_t is the m.s. integral of a process with continuous $R_N(\tau)$.

(d) $E[N_t] = E[E[N_t | \lambda]] = E[N_t | \lambda = 1] \cdot 1/3 + E[N_t | \lambda = 2] \cdot \frac{1}{3} + E[N_t | \lambda = 3] \frac{1}{3}$

$= (t + 2t + 3t)/3 = 2t$

$E[N_t^2] = E[E[N_t^2 | \lambda]] = (t^2 + t + 4t^2 + 2t + 9t^2 + 3t)/3 = \frac{14}{3} \cdot t^2 + 2t$

$Var[N_t] = E[N_t^2] - (E[N_t])^2\ 4 = \frac{2}{3} t^2 + 2t.$

Problem 4 (8/50, equally weighted parts)

A Gaussian random process X_t is wide-sense stationary, has zero mean and autocorrelation function $R_X(\tau) = 2e^{-|\tau|}$.

(a) Is X_t mean ergodic? Justify your answer.

(b) Find the MMSE estimate of X_3 given $X_1 = 1$.

Let another random process Y_t be defined as

$$Y_t = \int_0^t e^{-s} X_s ds \ .$$

(c) Compute the mean function of Y_t.

(d) Find $P[Y_3 \geq 2]$ and express it in terms of the function

$$Q(x) = \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \ .$$

(a) Yes, since $R_X(\tau) \to 0 \quad (|\tau| \to \infty)$

(b) For JGRVs, MMSE = LMMSE

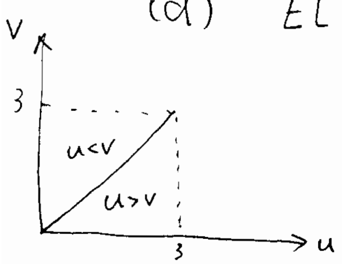
$$\hat{X}_{3,MMSE}(X_1 = 1) = E[X_3] + Cov(X_3, X_1)\, Cov^{-1}(X_1, X_1)\left(1 - E[X_1]\right)$$

$$= 0 + R_X(2) \cdot R_X^{-1}(0) \cdot 1$$

$$= 2e^{-2} \cdot 2^{-1} = e^{-2}$$

(c) $E[Y_t] = \int_0^t e^{-s} \cdot E[X_s]\, ds = 0$

(d) $E[Y_3^2] = \int_0^3 \int_0^3 e^{-u-v} E[X_u X_v]\, du dv$



$$= \int_0^3 \int_0^3 e^{-u-v} 2 \cdot e^{-|u-v|}\, du dv$$

$$= 2 \cdot 2 \cdot \int_0^3 du \int_0^u dv\ e^{-u-v} \cdot e^{v-u}$$

$$= 4 \cdot \int_0^3 u \cdot e^{-2u} du$$

$$= 1 - 7e^{-6}$$

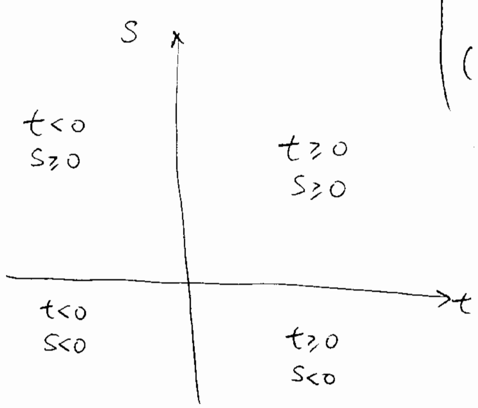
$Y_3 \sim N(0, 1-7e^{-6}) \quad \therefore P[Y_3 \geq 2] = P\left[\frac{Y_3}{\sqrt{1-7e^{-6}}} \geq \frac{2}{\sqrt{1-7e^{-6}}}\right] = Q\left(\frac{2}{\sqrt{1-7e^{-6}}}\right)$

Problem 5 (4/50)

Given a zero mean random process X_t with $R_X(t,s) = t^2 s^2 e^{-(|t|+|s|)}$ determine if its m.s. derivative $Y_t = X_t'$ exists. If so, determine $R_Y(t,s)$.

$$\frac{d}{ds} R_X(t,s) = \begin{cases} t^2 e^{-|t|} (2s e^{-s} - s^2 e^{-s}) & s \geq 0 \\ t^2 e^{-|t|} (2s e^{s} + s^2 e^{s}) & s < 0. \end{cases}$$

$$\frac{d}{dt}\frac{d}{ds} R_X(t,s) = \begin{cases} (2te^{-t} - t^2 e^{-t})(2se^{-s} - s^2 e^{-s}) & s \geq 0,\ t \geq 0 \\ (2te^{t} + t^2 e^{t})(2se^{-s} - s^2 e^{-s}) & s \geq 0 \quad t < 0 \\ (2te^{-t} - t^2 e^{-t})(2se^{s} + s^2 e^{s}) & s < 0,\ t \geq 0 \\ (2te^{t} + t^2 e^{t})(2se^{s} + s^2 e^{s}) & s < 0,\ t < 0. \end{cases}$$



We can see $\frac{d}{ds} R_X$, $\frac{d}{dt} R_X$, $\frac{d}{dt}\frac{d}{ds} R_X$ exist and continuous. (In particular, they are continuous at the boundaries $t=0$, or $s=0$)

$\therefore$ X_t is m.s. differentiable.

$$R_Y(t,s) = R_{X'X'}(t,s) = \frac{d}{dt}\frac{d}{ds} R_X(t,s).$$

Bonus Problem (5/50)

Let X_t be a zero-mean, wide-sense stationary random process with $R_X(\tau) = \sigma^2\delta(\tau)$ (i.e., X_t is *white*). Find h_t such that the random process

$$Y_t = \int_{-\infty}^{+\infty} X_s h_{t-s} ds$$

has autocorrelation function $K_Y(\tau) = e^{-\alpha|\tau|}$.

$$E[Y_t] = \int_{-\infty}^{+\infty} E[X_s]\, h_{t-s}\, ds = 0$$

$$\therefore K_Y(\tau) = E[Y_\tau \cdot Y_0^*]$$

$$= \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv\, E[X_u \cdot X_v^*]\, h(\tau-u)\, h^*(0-v)$$

$$= \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv \cdot \sigma^2 \delta(u-v)\, h(\tau-u)\, h^*(0-v)$$

$$= \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv' \cdot \sigma^2 \delta(v')\, h(\tau-u)\, h^*(v'-u) \quad \left(\begin{array}{c}\text{Change of Variable} \\ v' := u-v\end{array}\right)$$

$$= \sigma^2 \cdot \int_{-\infty}^{+\infty} h(\tau-u) \cdot h^*(-u)\, du$$

$$= \sigma^2 \cdot \big(h(u) * h^*(-u)\big)(\tau)$$

$$\therefore \mathcal{F}(K_Y(\tau)) = \sigma^2 \cdot H(\omega) \cdot H^*(\omega)$$

$$\because \mathcal{F}(e^{-\alpha|\tau|}) = \frac{2\alpha}{\alpha^2+w^2} = \frac{\sqrt{2\alpha}}{\alpha+jw} \cdot \frac{\sqrt{2\alpha}}{\alpha-jw} = \sigma^2 \cdot H(w) \cdot H^*(w).$$

One possible solution is $H(w) = \dfrac{\sqrt{2\alpha}}{\sigma(\alpha+jw)}$

$$\therefore h(t) = \begin{cases} \dfrac{\sqrt{2\alpha}}{\sigma} e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$