ECE 534 **Fall 2009**

December 16, 2009

Solutions to Final Exam

1. (24 pts, equally weighted parts) *True or False.*

   (a) If $U_1, U_2, \ldots,$ is a sequence i.i.d. Unif[0,1] random variables and $X_n = (U_n)^n$, $n \geq 1$, then X_n converges in probability as $n \to \infty$.

   Ans: True. In fact $X_n \overset{m.s.}{\to} 0$, since $\mathsf{E}[X_n^2] = \mathsf{E}[U_n^{2n}] = 1/(2n+1) \to 0$ as $n \to \infty$.

   (b) Suppose $\mathsf{E}[X_n^2] < \infty$, for all n. If $X_n \overset{p.}{\to} c$, where c is a deterministic constant, then $X_n \overset{m.s.}{\to} c$ as well.

   Ans: False. Consider $\Omega = [0,1]$ with the uniform probability measure, and let $X_n = n\mathbb{1}_{\{\omega \in [0,1/n]\}}$. Then $X_n \overset{a.s.}{\to} 0$ and hence $X_n \overset{p.}{\to} 0$, but $\mathsf{E}[X_n^2] = n \to \infty$ as $n \to \infty$.

   (c) If $(X_t, t \in \mathbb{R})$ is Gaussian random process with covariance function $C_X(s,t) = st + \min\{s,t\}$, then (X_t) *cannot* be a Markov process.

   Ans: False. A Gauss-Markov process needs to satisfy, for $r < s < t$

   $$C_X(r,t) = \frac{C_X(r,s)\, C_X(s,t)}{C_X(s,s)}$$

   It is easy to check that the given covariance function does satisfy the condition and is indeed Markov.

   (d) If X and Y are jointly Gaussian random variables with finite second moments, then

   $$\mathsf{E}[(X - \mathsf{E}[X|Y])^2] = \mathsf{E}[(X - \hat{\mathsf{E}}[X|Y, Y^2])^2]$$

   Ans: True. Since X and Y are jointly Gaussian, the MMSE estimate is linear. So adding a quadratic term to the LMMSE estimator cannot decrease the MSE below that obtained by just having the linear term.

   (e) The function $R(\tau) = |\sin(\tau)|$ is a valid auto-correlation function for a WSS process.

   Ans: False. $R(0) = 0 < R(\pi/2) = 1$.

   (f) The function $S(\omega) = e^{-|\omega|}|\sin(\omega)|$ is a valid power spectral density for a WSS process.

   Ans: True. Since $S(\omega)$ is real-valued and ≥ 0 for all ω.

   (g) A time-homogenous discrete-state Markov process (X_t) satisfies $\underline{\pi}(t) = \underline{\pi}$ for some distribution $\underline{\pi}$. Then (X_t) must be a (strictly) stationary process.

   Ans: True. For any n and $t_1 < t_2 < \cdots < t_n$, the joint distribution of $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ depends on the marginal of X_{t_1} and the transition matrices $H(t_1,t_2)$, $H(t_2,t_3)$, $\ldots H(t_{n-1},t_n)$, all of which are invariant if we replace t_i by $t_i + \tau$, $i = 1, 2, \ldots, n$.

   (h) For zero-mean jointly WSS (X_t) and (Y_t), the noncausal Wiener filter for optimum linear estimation of X_t given $\{Y_s : s \in \mathbb{R}\}$ is necessarily *time-invariant*.

   Ans: True. It is easy to see that the linear Kernel $h(u,v)$ for optimum estimation of X_t given $\{Y_s : s \in \mathbb{R}\}$ must be the same that for estimation of $X_{t+\tau}$ from $\{Y_s : s \in \mathbb{R}\} = \{Y_{s+\tau} : s \in \mathbb{R}\}$, which means that $h(u,v) = h(u+\tau, v+\tau)$ for all $\tau \in \mathbb{R}$.

2. (12 pts) *CLT and Chernoff Bound.* Let $\{X_k : k \geq 0\}$ be a sequence of i.i.d. random variables with

$$\mathsf{P}\{X_k = -1\} = \frac{1}{4} \quad \mathsf{P}\{X_k = 0\} = \frac{1}{2} \quad \mathsf{P}\{X_k = 1\} = \frac{1}{4}$$

Suppose $S_n = \sum_{k=1}^{n} X_k$.

(a) Find $M_X(\theta)$, the moment generating function of X_k.
Ans: $M_X(\theta) = \mathsf{E}[e^{\theta X_n}] = \frac{1}{4}(e^{\theta} + e^{-\theta}) + \frac{1}{2}$.

(b) Use the Central Limit Theorem to find an approximation for $\mathsf{P}\{S_{100} \geq 50\}$ in terms of the $Q(\cdot)$ function.
Ans: $\mu = \mathsf{E}[X_n] = 0$ and $\sigma^2 = \text{Var}(X_n) = \mathsf{E}[X_n^2] = \frac{1}{2}$. Thus, by the Central Limit Theorem, $(S_{100}/(10\sigma)$ is approximately $\mathcal{N}(0,1)$. Therefore,

$$\mathsf{P}\{S_{100} > 50\} = \mathsf{P}\left\{\frac{S_{100}}{10\sigma} > \frac{50}{10\sigma}\right\} \approx Q\left(5\sqrt{2}\right)$$

(c) Now use the Chernoff Bound to find a bound on $\mathsf{P}\{S_{100} \geq 50\}$.
Ans: By the Chernoff Bound,

$$\mathsf{P}\{S_{100} \geq 50\} = \mathsf{P}\left\{\frac{S_{100}}{100} \geq \frac{1}{2}\right\} \leq e^{-100\,\ell(0.5)}$$

where $\ell(0.5)$ is obtained by maximizing

$$0.5\,\theta - \ln M_X(\theta) = 0.5\theta - \ln(2 + e^{\theta} + e^{-\theta}) + \ln(4)$$

Taking the derivative and setting it equal to zero, we obtain that the optimizing θ^* satisfies

$$0.5 = \frac{e^{\theta^*} - e^{-\theta^*}}{2 + e^{\theta^*} + e^{-\theta^*}}$$

Setting $x = e^{\theta^*}$ reduces the above equation to the quadratic $x^2 - 2x - 3 = 0$, which has the solutions $x = 3$ and $x = -1$. Since x has to be positive, we conclude that $x = 3 \implies \theta^* = \ln 3$. Thus $\ell(0.5) = 0.5 \ln 3 - \ln(4/3) = \ln(3\sqrt{3}/4)$. Therefore,

$$\mathsf{P}\{S_{100} \geq 50\} \leq \left(\frac{3\sqrt{3}}{4}\right)^{-100}$$

3. (14 pts) *Linear Innovations.* Let $(Y_k : k \geq 1)$ be a discrete-time *zero-mean* WSS random process with ACF

$$R_Y(k) = (0.5)^{|k|}$$

(a) Find the linear innovations sequence $\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3$ corresponding to the first three samples of the process Y_1, Y_2, Y_3.
Ans: $\tilde{Y}_1 = Y_1$, and $\tilde{Y}_2 = Y_2 - \hat{\mathsf{E}}[Y_2|\tilde{Y}_1] = Y_2 - \hat{\mathsf{E}}[Y_2|Y_1]$. Now, $\text{Var}(Y_1) = 1$ and $\text{Cov}(Y_2, Y_1) = 0.5$. Thus $\hat{\mathsf{E}}[Y_2|Y_1] = 0.5Y_1$, and $\tilde{Y}_2 = Y_2 - 0.5Y_1$. Now by linear innovations applied recursively, $\tilde{Y}_3 = Y_3 - (\hat{\mathsf{E}}[Y_3|\tilde{Y}_2] + \hat{\mathsf{E}}[Y_3|\tilde{Y}_1])$. Since $\text{Cov}(Y_3, Y_1) = 0.25$, $\hat{\mathsf{E}}[Y_3|\tilde{Y}_1] = \hat{\mathsf{E}}[Y_3|Y_1] = 0.25Y_1$. Furthermore, $\text{Var}(\tilde{Y}_2) = \text{Var}(Y_2) + 0.25\text{Var}(Y_1) - \mathsf{E}[Y_1 Y_2] = \frac{3}{4}$, and $\text{Cov}(Y_3, \tilde{Y}_2) = \mathsf{E}[Y_3 Y_2] - 0.5\mathsf{E}[Y_3 Y_1] = 0.5 - 0.125 = \frac{3}{8}$, which means that $\hat{\mathsf{E}}[Y_3|\tilde{Y}_2] = \frac{3}{8}\frac{4}{3}\tilde{Y}_2 = 0.5\tilde{Y}_2$. Thus $\tilde{Y}_3 = Y_3 - 0.25Y_1 - 0.5(Y_2 - 0.5Y_1) = Y_3 - 0.5Y_2$.

(b) Now suppose X is a *zero mean* random variable with finite second moment satisfying

$$\mathsf{E}[XY_1] = 1, \quad \mathsf{E}[XY_2] = 0.5, \quad \mathsf{E}[XY_3] = 0.25$$

Find the LMMSE estimate $\hat{\mathsf{E}}[X|Y_1, Y_2, Y_3]$.

Ans: $\hat{\mathsf{E}}[X|Y_1, Y_2, Y_3] = \hat{\mathsf{E}}[X|\tilde{Y}_1] + \hat{\mathsf{E}}[X|\tilde{Y}_2] + \hat{\mathsf{E}}[X|\tilde{Y}_3]$. Now, $\hat{\mathsf{E}}[X|\tilde{Y}_1] = \hat{\mathsf{E}}[X|Y_1]\mathsf{E}[XY_1]\mathrm{Var}(Y_1)^{-1}Y_1 = Y_1$. Furthermore, it is easy to see that $\mathsf{E}[X\tilde{Y}_2] = \mathsf{E}[X\tilde{Y}_3] = 0$, which means that $\hat{\mathsf{E}}[X|\tilde{Y}_2] = \hat{\mathsf{E}}[X|\tilde{Y}_3] = 0$. Thus $\hat{\mathsf{E}}[X|Y_1, Y_2, Y_3] = Y_1$.

4. (16 pts) *Poisson process.* Let $(N_t : t \geq 0)$ be a Poisson process with parameter $\lambda = 1$.

(a) Find $\mathsf{P}\{N_3 \leq 2 \mid N_1 \geq 1\}$.

Ans:

$$\mathsf{P}\{N_3 \leq 2 \mid N_1 \geq 1\} = \frac{\mathsf{P}\{N_3 \leq 2\ ,\ N_1 \geq 1\}}{\mathsf{P}\{N_1 \geq 1\}}$$

Now, $\mathsf{P}\{N_1 \geq 1\} = 1 - \mathsf{P}\{N_1 = 0\} = 1 - e^{-1}$, and using the independent increment property of (N_t),

$$\mathsf{P}\{N_3 \leq 2\ ,\ N_1 \geq 1\} = \mathsf{P}\{N_1 = 2\ ,\ N_3 - N_1 = 0\} + \mathsf{P}\{N_1 = 1\ ,\ N_3 - N_1 \leq 1\} = \cdots = \frac{7}{2}e^{-3}$$

Thus $\mathsf{P}\{N_3 \leq 2 \mid N_1 \geq 1\} = \frac{7}{2}\frac{e^{-3}}{1-e^{-1}}$

(b) Find $\mathsf{P}\{N_1 \geq 1 \mid N_3 \leq 2\}$.

Ans: $\mathsf{P}\{N_3 \leq 2\} = e^{-3} + 3e^{-3} + \frac{9}{2}e^{-3} = \frac{17}{2}e^{-3}$. Thus $\mathsf{P}\{N_1 \geq 1 \mid N_3 \leq 2\} = \frac{7}{17}$.

(c) Now suppose we define the random variable Z via the m.s. integral

$$Z = \int_0^1 N_t dt$$

Find the LMMSE estimate $\hat{\mathsf{E}}[N_2|Z]$.

Ans: The autocovariance function of (N_t) is given by $C_N(s,t) = \min(s,t)$.

$$\mathsf{E}[Z] = \int_{t=0}^1 t dt = \frac{1}{2}, \quad \mathrm{Var}(Z) = \int_0^1 \int_0^1 C_N(s,t) dt ds = \int_0^1 \int_0^1 \min(s,t)\ dt ds = \frac{1}{3}$$

Furthermore,

$$\mathrm{Cov}(N_2, Z) = \int_{t=0}^1 C_N(t,2) dt = \int_{t=0}^1 t dt = \frac{1}{2}$$

Thus $\hat{\mathsf{E}}[N_2|Z] = 2 + \frac{1}{2}\, 3\, (Z - \frac{1}{2}) = \frac{3}{2}Z + \frac{5}{4}$.

5. (20 pts) *FSMP.* Consider a time-homogeneous discrete-time Markov process $(X_k : k \geq 0)$ with state space $\mathcal{S} = \{-1, 0, 1\}$ and one-step probability transition matrix P given by

$$P = \begin{bmatrix} 0.2 & 0.8 & 0 \\ 0.4 & 0.2 & 0.4 \\ 0 & 0.8 & 0.2 \end{bmatrix}$$

(a) Find the equilibrium distribution $\underline{\pi}$.

Ans: Using the fact that $\underline{\pi} = \underline{\pi}P$ and $\underline{\pi}\ \underline{e} = 1$, it is easy to show that $\pi_{-1} = \pi_1 = \frac{1}{4}$ and $\pi_0 = \frac{1}{2}$.

For the remaining parts, assume that X_0 has the equilibrium distribution.

(b) Determine whether or not (X_k) is a martingale.

Ans: No. For example, $\mathsf{E}[X_2|X_1 = -1] = (0.2)(-1) + (0.8)(0) = -0.2 \neq -1$.

(c) Find the joint distribution of X_1 and X_2. (You may want to put the values in a table.)

Ans: $P\{X_2 = j, X_1 = i\} = \pi_i P_{i,j}$. Thus the joint pmf is described by table

	-1	0	1
-1	0.05	0.2	0
0	0.2	00.1	0.2
1	0	0.2	0.05

(d) Let the discrete-time process $(Y_k : k \geq 0)$ be defined by

$$Y_k = X_1 + kX_2, \quad k \geq 0$$

Find the mean and autocorrelation function of (Y_k).

Ans: $\mathsf{E}[X_1] = \mathsf{E}[X_2] = 0$, $\mathsf{E}[X_1^2] = \mathsf{E}[X_2^2] = \frac{1}{2}$, and $\mathsf{E}[X_1X_2] = (-1)(-1)(0.05) + (1)(1)(0.05) = 0.1$. Thus

$$\mathsf{E}[Y_k] = 0, \quad R_Y(k, m) = \mathsf{E}[Y_kY_m] = \frac{1}{2} + \frac{km}{2} + (0.1)(k + m)$$

(e) Find $\mathsf{E}[Y_2|Y_1, Y_0]$.

Ans: $Y_2 = X_1 + 2X_2 = 2Y_1 - Y_0$. Thus $\mathsf{E}[Y_2|Y_1, Y_0] = 2Y_1 - Y_0$.

(f) Determine whether or not (Y_k) is a Markov process.

Ans: No, since $\mathsf{E}[Y_2|Y_1, Y_0]$ depends on both Y_1 and Y_0. In particular

$$\mathsf{E}[Y_2|Y_1 = 1, Y_0 = 1] = 1 \neq \mathsf{E}[Y_2|Y_1 = 1] = 1 + \mathsf{E}[X_2|Y_1 = 1] = 1 + \frac{1}{2}$$

6. (14 pts) *Filtering.* Consider a zero-mean WSS process (X_t) with autocorrelation function

$$R_X(\tau) = \frac{1}{2}e^{-|\tau|}$$

Suppose (X_t) is passed through a linear time-invariant system with transfer function

$$H(\omega) = \frac{1}{3 + j\omega}$$

to produce the output process (Y_t).

(a) Find $S_{YX}(\omega)$ and use it to find $R_{YX}(\tau)$.

Ans:

$$S_{YX}(\omega) = H(\omega)S_X(\omega) = \frac{1}{3 + j\omega}\frac{1}{1 + \omega^2} = \frac{1}{4}\frac{1}{1 + j\omega} + \frac{1}{8}\frac{1}{1 - j\omega} - \frac{1}{8}\frac{1}{3 + j\omega}$$

where the last equality follows from using partial fractions. Applying the inverse Fourier transform

$$R_{YX}(\tau) = \left(\frac{1}{4}e^{-\tau} - \frac{1}{8}e^{-3\tau}\right)\mathbb{1}_{\{\tau \geq 0\}} + \frac{1}{8}e^{\tau}\mathbb{1}_{\{\tau < 0\}}$$

(b) Find $S_Y(\omega)$ and use it to find $R_Y(\tau)$.

Ans: $S_Y(\omega) = S_X(\omega)|H(\omega)|^2$. Using the Fourier transform pairs given to you

$$S_Y(\omega) = \frac{1}{9+\omega^2}\,\frac{1}{1+\omega^2} = \frac{1}{8}\left[\frac{1}{1+\omega^2} - \frac{1}{9+\omega^2}\right] = \frac{1}{8}\left[\frac{1}{2}\,\frac{2}{1+\omega^2} - \frac{1}{6}\,\frac{6}{9+\omega^2}\right]$$

and therefore

$$R_Y(\tau) = \frac{1}{16}e^{-|\tau|} - \frac{1}{48}e^{-3|\tau|}$$

(c) Find the LMMSE estimate $\hat{\mathsf{E}}[X_2|Y_1]$.

Ans: $\mathsf{E}[X_2Y_1] = \mathsf{E}[Y_1X_2] = R_{YX}(-1) = \frac{1}{8}e^{-1}$ and $\mathrm{Var}(Y_1) = R_Y(0) = \frac{1}{24}$. Thus

$$\hat{\mathsf{E}}[X_2|Y_1] = 0 + \frac{1}{8}e^{-1}24(Y_1 - 0) = 3e^{-1}Y_1$$

7. (Extra credit – attempt only if you have time; I will not grade your answer if you have not finished the rest of the exam)

The Cliff-Hanger. A drunken man is near a cliff. From where he stands, one step toward the cliff would send him over the edge. He takes a random step either towards or away from the cliff. At any step, his probability of taking a step away from the cliff is p, and of a step towards the cliff is $(1-p)$. Find the probability that he will escape unharmed as a function of p, for the entire range $0 \le p \le 1$.

Ans: This is essentially the Gambler's ruin problem with initial wealth of $k = 1$ and goal of $b = \infty$. It is easier to calculate the probability that the man will fall off the cliff, which we denote by ρ. Using the formula we derived in class, we get (for $p \neq \frac{1}{2}$)

$$\rho = \lim_{b\to\infty} \frac{\left(\frac{1-p}{p}\right) - \left(\frac{1-p}{p}\right)^b}{1 - \left(\frac{1-p}{p}\right)^b}$$

If $0 \le p < \frac{1}{2}$, $\left(\frac{1-p}{p}\right)^b$ converges to ∞ as $b \to \infty$, which means that $\rho = 1$.

If $\frac{1}{2} < p \le 1$, $\left(\frac{1-p}{p}\right)^b$ converges to 0 as $b \to \infty$, which means that $\rho = \frac{1-p}{p}$.

For $p = \frac{1}{2}$, we use the boundary conditions to get $\rho = \lim_{b\to\infty} 1 - \frac{1}{b} = 1$.

We can also solve the problem directly without using the Gambler's ruin solution. Note that the probability of falling off the cliff starting two steps away is simply ρ^2. Thus $\rho = (1-p) + \rho^2 p$, which we can solve to get $\rho = 1$ or $\rho = (1-p)/p$. If $p < \frac{1}{2}$, the second solution is impossible since ρ has to be ≤ 1. For $p = 1$, it is clear that $\rho = 0$. Now, we can argue that ρ should be continuous in p to conclude that for $p \ge \frac{1}{2}$, $\rho = (1-p)/p$.