Solutions to Practice final

ECS 534
spring 2005

Problem 1

a) False $\qquad \begin{vmatrix} 2 & r \\ r & 3 \end{vmatrix} \geq 0 \Rightarrow |r| \leq \sqrt{6}$

b). True. $\qquad E[Z_t] = \mu_x \mu_y$

$$cov_Z(t,s) = cov_X(t-s)\, cov_Y(t-s)$$

c) False $\qquad S_X(\omega) \leq 0 \quad \forall \omega \geq 1.$

PSD has always be positive.

d) True.

e) True.

f) True.

g) False. $\tan^{-1}(\cdot)$ is a concave function, thus it won't satisfy Jensen's Inequality.

h) True

i) False $\qquad |R_X(t,s)| \leq \sqrt{R_X(t,t) R_X(s,s)}$

j). False $\qquad R_X(\tau) \leq R_X(0), \ \forall \tau \neq 0.$ The slope @ $\tau = 0$ has to be 0.

k) False $\qquad 2R_{XY}(\tau) \overset{?}{\leq} R_X(0) R_Y(0).$ No.

l) ~~So~~ True $\qquad S_W(0) = \frac{4}{2} = \int_{-\infty}^{\infty} R_X(\tau) d\tau \leq \pi$

Problem 2

a) $X = |Y|$

$\hat{X}_{MMSE} = E[X|Y] = |Y|$

$MSE_1 = 0$

b). $\hat{X}_{LMMSE} = E[X] + \frac{cov(X,Y)}{cov(Y,Y)}(Y - E[Y])$

$E[Y] = 0$

$E[X] = E[|Y|] = 0.8$

$cov(Y,Y) = 1$

$cov(X,Y) = E[XY] = E[|Y|Y] = 0.$

$\hat{X}_{LMMSE} = 0.8$

$MSE_2 = E[(X-0.8)^2] = E[X^2] - 1.6\, E[X] + 0.64 = 0.32$

c). $\hat{X}(y) = a + by + cy^2$

$MSE = E[[X - (a + by + cy^2)]^2]$

$0 = \frac{\partial}{\partial a} E[\quad]$

$0 = \frac{\partial}{\partial b} E[\quad]$

$0 = \frac{\partial}{\partial c} E[\quad]$

$\Rightarrow \quad a = c = 0.4, \quad b = 0.$

$\hat{X}(y) = 0.4 + 0.4\, y^2$

Problem 3 $\quad Y_n = \frac{1}{\sqrt{n}} X_n - 9\sqrt{n}$.

A
$$= \frac{1}{\sqrt{n}} (\underbrace{X_n - 9n}_{Z_n}) = \frac{1}{\sqrt{n}} Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i - Z_{i-1}$$

$(Z_0 = X_0 = 0)$.

Because X_n is Poisson w/ independent increments, $\{Z_i - Z_{i-1}\}_{i=1}^{n}$ are iid. Each term has 0-mean, and variance $(Z_i - Z_{i-1}) = \lambda = 9$.

By Central Limit Theorem, Y_n converges to a Gaussian.
$$\lim_{n\to\infty} Y_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i - Z_{i-1} \xrightarrow{d} N(0, 9).$$

B) $\quad E[W_n | W_{n-1}, W_{n-2}, \ldots, W_1] = W_{n-1}$

$$E[\alpha \cdot W_{n-1} \cdot X_n] = \alpha W_{n-1} \cdot E[X_n] = W_{n-1}$$
$$\alpha \cdot E[X_n] = 1$$
$$\alpha = \frac{1}{E[X_n]} = \frac{1}{4}.$$

Problem 4. A. $R_X(\tau) = 2e^{-|\tau|} - e^{-2|\tau|}$

$$R_X(t-s) = 2e^{-(t-s)} - e^{-2(t-s)} \qquad \tau \geq 0 \Rightarrow t \geq s.$$
$$= 2e^{+(t-s)} - e^{2(t-s)} \qquad t < s.$$

$$\frac{\partial}{\partial s} R_X(t-s) = 2e^{-(t-s)} - 2e^{-2(t-s)} \qquad t \geq s$$
$$= -2e^{(t-s)} + 2e^{2(t-s)} \qquad t < s.$$

The first partial is continuous @ $t = s$. ✓

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} R_X(t-s) = -2e^{-(t-s)} + 4e^{-2(t-s)} \qquad t \geq s$$
$$= -2e^{t-s} + 4e^{2(t-s)} \qquad t < s.$$

The 2nd mixed partial is also continuous @ $t = s$

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\frac{\partial}{\partial s} R_X(t-s) = -2e^{-(t-s)} + 8e^{-2(t-s)}. \qquad t \geq s$$
$$= 2e^{t-s} - 8e^{2(t-s)} \qquad t < s.$$

$$\Big|_{t=s} = \begin{array}{ll} 8-2 = 6 & t \geq s \\ 2-8 = -6 & t < s. \end{array}$$

The 3rd mixed partial has a discontinuity @ $t = s$.
$\Rightarrow$ doesn't exist!

Since X is WSS, $R_X(\tau)$, $R_X'(\tau)$ & $R_X''(\tau)$ exist & continuous, then X_t' exists in the mean-square sense.

$\Rightarrow k = 1$.

Part B. X_0 & $X_{\frac{\pi}{2}}$ are jointly Gaussian.

$$\mu_{X_0} = 1 \qquad \sigma_{X_0}^2 = 4$$

$$\mu_{X_{\frac{\pi}{2}}} = 0 \qquad \sigma_{X_{\frac{\pi}{2}}}^2 = 4 \qquad \mathrm{cov}(X_0, X_{\frac{\pi}{2}}) = E\{X_0 X_{\frac{\pi}{2}}\} = 4e^{-\frac{\pi}{2}}$$

$$\begin{bmatrix} X_0 \\ X_{\frac{\pi}{2}} \end{bmatrix} \sim N\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 & 4e^{-\frac{\pi}{2}} \\ 4e^{-\frac{\pi}{2}} & 4 \end{bmatrix} \right)$$

$$f_{X_{\frac{\pi}{2}} | X_0}(x | X_0 = 1) = \frac{f_{X_{\frac{\pi}{2}}, X_0}(x_{\frac{\pi}{2}}, x_0)}{f_{X_0}(x_0)}.$$

$$\sim N\left(e^{-\frac{\pi}{2}}, 4(1 - e^{-\pi})\right).$$

Problem 5.

a) $\phi_1(t) = s(t)$.

$$Y_t = \sum_{i=1}^{\infty} (Y_t, \phi_i(t))\, \phi_i(t)$$

$$= (Y_t, \phi_1(t))\, \phi_1(t) + \sum_{i=2}^{\infty} (Y_t, \phi_i(t))\, \phi_i(t)$$

$$(Y_t, \phi_1(t)) = \int (X s(t) + N_t)\, \overline{s(t)}\, dt = X \int \|s(t)\|^2 dt + \int N_t\, \overline{s(t)}\, dt$$

$$= X + n_1 \quad , \text{ where } n_1 \sim \mathcal{N}(0, \sigma^2)$$

$i \geq 2$ $\quad (Y_t, \phi_i(t)) = \int (X s(t) + N_t)\, \overline{\phi_i(t)}\, dt \quad , \quad s(t) = \phi_1(t) \perp \phi_i(t),\ i \geq 2.$

$$= X \cancel{\int s(t)\, \phi_i(t)\, dt}^{\to 0} + n_i$$

Sufficient Statistics of Y_t
$$\begin{cases} (Y_t, \phi_1(t)) = X + n_1 \\ (Y_t, \phi_i(t)) = n_i \quad , \quad i = 2, 3, 4 \cdots\cdots \end{cases}$$

b)
$$\left.\begin{aligned} \hat{X}_{MMSE} &= E[X \mid Y_t,\ 0 \leq t \leq T] \\ &= E[X \mid X + n_1, n_2, n_3 \cdots] \end{aligned}\right\} \text{By K-L Expansion of part a).}$$
$$= E[X \mid X + n_1]$$

$$\hat{X}_{MMSE} = \frac{1}{1 + \sigma^2} (Y_t, \phi_1(t))$$

Problem 6 $\quad S_X(\omega) = \dfrac{2}{1+\omega^2}$.

a) $$S_Y(\omega) = S_X(\omega)\,|H(\omega)|^2 = \left(\frac{2}{1+\omega^2}\right)\left(\frac{2b}{b^2+\omega^2}\right).$$

b). $$R_{YX}(\tau) = R_X(\tau) * h(\tau).$$
$$S_{YX}(\omega) = S_X(\omega)\,H(\omega) = \cancel{\left(\frac{2}{1+\omega^2}\right)}\left(\frac{1}{1+j\omega} - \frac{2}{2+j\omega} + \frac{1}{3+j\omega}\right)$$

$$H(\omega) = \frac{S_{YX}(\omega)}{S_X(\omega)}$$

It's unique, because the system is stable & causal.

Problem 7. a)

$$S_X(\omega) = \frac{2\lambda}{\lambda^2+\omega^2}.$$

$$S_Y(\omega) = S_N(\omega) + S_X(\omega)\,|G(\omega)|^2.$$

$$= \sigma^2 + \frac{2\lambda}{\lambda^2+\omega^2}\,\frac{\pi}{\alpha}\,e^{-\frac{\omega^2}{2\alpha}}.$$

$$S_{XY}(\omega) = S_X(\omega)\,\overline{G(\omega)}$$
$$= \frac{2\lambda}{\lambda^2+\omega^2}\sqrt{\frac{\pi}{\alpha}}\,e^{-\frac{\omega^2}{2\alpha}}.$$

$$H(\omega) = \frac{S_{XY}(\omega)}{S_Y(\omega)}$$

Problem 8.

a).

$$\left[X(\omega) + Y(\omega)\, H_2(\omega)\right] H_1(\omega) + N(\omega) = Y(\omega).$$

$$G_1(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{H_1(\omega)}{1 - H_1(\omega)\, H_2(\omega)} = \frac{2}{1 - \frac{4}{5+j\omega}} = \frac{2(5+j\omega)}{1+j\omega}.$$

$$G_2(\omega) = \frac{Y(\omega)}{N(\omega)} = \frac{1}{1 - H_1(\omega) H_2(\omega)} = \frac{1}{1 - \frac{4}{5+j\omega}} = \frac{5+j\omega}{1+j\omega}.$$

$$S_y(\omega) = S_x(\omega)\, |G_1(\omega)|^2 + S_N(\omega)\, |G_2(\omega)|^2.$$

$$S_{xy}(\omega) = S_x(\omega)\, \overline{G_1(\omega)}.$$

$$H(\omega) = \frac{S_{xy}(\omega)}{S_y(\omega)}.$$