University of Illinois at Urbana-Champaign
Department of Electrical and Computer Engineering

ECE 434: Random Processes

Spring 2004

Final Exam

Friday, May 7, 1:30–4:30pm, 106B1 Engineering Hall

READ THESE COMMENTS BEFORE STARTING THE EXAM!

- This is a **closed-book** exam! You are allowed three sheets of *handwritten* notes (both sides). Calculators should not be necessary, but feel free to use one.
- **Write your name on the answer booklet.**
- There are **eight unequally weighted** problems for a total of **60 points**. A **bonus** problem worth **5 points** is also included. Problems are *not* necessarily in order of difficulty.
- A correct answer does not guarantee credit; an incorrect answer does not guarantee loss of credit. **Provide clear explanations, show all relevant work and justify your answers!** If we cannot make sense of your writing or reasoning, you may loose points.
- Read each problem carefully and *think* before performing detailed calculations.
- Only the supplied answer booklet is to be handed in. **No additional pages will be considered in the grading.** You may want to work things through in the blank areas of the exam and then neatly transfer to the answer sheet the work you would like us to look at.

Formulas:

- $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$, $\quad \sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$
- $\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$, $\quad \cos a \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$
- $\sin a \cos b = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]$, $\quad \cos(2\theta) = \cos^2\theta - \sin^2\theta$
- $\mathcal{FT}\{e^{-\alpha t}u(t)\} = \frac{1}{\alpha + j2\pi f}$, $\quad \alpha > 0$
- $\mathcal{FT}\{e^{-\alpha t^2}\} = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\omega^2}{4\alpha}}$

Problem 1 (12/60, equally weighted parts)

This problem has **twelve independent** true/false questions.

(a) If the matrix $\begin{bmatrix} 2 & r \\ r & 3 \end{bmatrix}$ is the covariance matrix of a zero-mean random vector $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, then r necessarily satisfies $|r| \leq 2$.

(b) If X_t and Y_t are independent wide-sense stationary (WSS) random processes, then $Z_t = X_t \cdot Y_t$ is a WSS random process.

(c) The function $S_X(\omega) = \dfrac{1-\omega^2}{1+\omega^2}$ is a valid power spectral density (PSD) for a wide-sense stationary random process X_t.

(d) If $A \subset B$ and $P[B] \neq 0$, then $P[A|B] \geq P[A]$.

(e) If X and Y are i.i.d. Gaussian random variables with mean 0 and variance 1, then $Z_1 = X - Y$ and $Z_2 = X + Y$ are independent, identically distributed (i.i.d.) Gaussian random variables.

(f) Let the random process Y_t be defined as $Y_t = X_1 \sin(2\pi t) + X_2 \cos(2\pi t)$, where X_1 and X_2 are zero-mean random variables that satisfy $E[X_1 X_2] = 0$ and $E[X_1^2] = E[X_2^2] = \sigma^2$. Then, Y_t is a wide-sense stationary random process.

(g) If X is a nonnegative random variable, then $\arctan(E[X]) \le E[\arctan(X)]$.

(h) For any two random variables X and Y, it is true that $E[E[X|Y,Z]|Y] = E[X|Y]$.

(i) The function $R_X(t,s) = e^{t-s}$ *cannot* be the autocorrelation function of a random process X_t.

(j) If $R_X(\tau)$ is the autocorrelation function of a mean square (m.s.) differentiable wide-sense (WSS) random process, then it is possible that

$$R'(\tau) = 1 + \sin^2(\tau^2) \ .$$

(k) It is possible for two jointly wide-sense stationary (jointly WSS) random processes X_t and Y_t to satisfy

$$\begin{aligned} R_X(\tau) &= 2e^{-3|\tau|} \ , \\ R_Y(\tau) &= 5e^{-|\tau|} \ , \\ R_{XY}(\tau) &= 15e^{-3|\tau|} - 2e^{-|\tau|} \ . \end{aligned}$$

(l) The autocorrelation function $R_X(\tau)$ of a wide-sense stationary (WSS) random process N_t with power spectral density (PSD) $S_N(\omega) = \frac{4}{2+\omega^4}$ satisfies

$$\int_{-\infty}^{+\infty} R_X(\tau) d\tau \le \pi \ .$$

Problem 2 (6/60, equally weighted parts)

Suppose that the random variable Y is Gaussian with mean 0 and variance 1. We know that random variable X satisfies $X = |Y|$ and we are interested in estimating X based on the observation that $Y = y$. (**Hint:** To answer the following questions, you may want to use the following numerical values: $E[|Y|] = 0.8$, $E[|Y|^3] = 1.6$, $E[|Y|^4] = 3$.)

(a) Find the minimum mean square error (MMSE) estimator $\widehat{X}_{MMSE}(y)$? What is the associated mean square error MSE_1?

(b) Find the *linear* minimum mean square error (LMMSE) estimator $\widehat{X}_{LMMSE}(y) = \alpha + \beta y$? What is the associated mean square error MSE_2?

(c) Find the minimum mean square error estimator of the form

$$\widehat{X}(y) = a + by + cy^2 \ .$$

Problem 3 (8/60, equally weighted parts)

This problem has two *independent* parts.

Part A: Let X_t, $0 \leq t < +\infty$, be a Poisson random process with $X_0 = 0$ and rate $\lambda = 9$. Let

$$Y_n = \frac{1}{\sqrt{n}} X_n - 9\sqrt{n}$$

for $n = 1, 2, 3, ...$ To what random variable, if any, does the sequence Y_n converge in distribution (i.d.)?

Part B: Let X_n for $n = 1, 2, 3, ...$ be a sequence of independent, identically distributed (i.i.d.) Poisson random variables with parameter $\lambda = 4$. Let

$$W_n = \alpha^n X_1 X_2 X_3 ... X_n$$

for $n = 1, 2, 3, ...$ For what values of α, if any, is W_n a Martingale?

Problem 4 (6/60, equally weighted parts)

This problem has two *independent* parts.

Part A: A wide-sense stationary random process X_t has mean 0 and autocorrelation function

$$R_X(\tau) = 2e^{-|\tau|} - e^{-2|\tau|} \; .$$

Does the mean square derivative $\frac{d^k}{dt^k} X_t$ exists as a well-defined second order process for all k? If not, what is the highest value of k for which it exists?

Part B: A Gaussian random process X_t satisfies

$$\begin{aligned} \mu_X(t) &= \cos(t) \; , \\ R_X(t, s) &= \cos(t)\cos(s) + 4e^{-|t-s|} \; . \end{aligned}$$

Find the conditional density of $X_{\frac{\pi}{2}}$ given $X_0 = 1$, i.e., find

$$f_{X_{\frac{\pi}{2}}|X_0=1}(x|X_0 = 1) \; .$$

Problem 5 (8/60, equally weighted parts)

Consider the following estimation problem: a Gaussian random variable X with mean 0 and variance 1 is used to modulate the amplitude of a *deterministic* signal $s(t)$, $0 \leq t \leq T$, that satisfies

$$\int_0^T |s(t)|^2 dt = 1 \ .$$

The modulated waveform is corrupted by an additive white Gaussian noise process N_t with mean 0 and autocorrelation function $R_N(\tau) = \sigma^2 \delta(\tau)$. Assume that N_t and X are independent. The resulting random process Y_t satisfies

$$Y_t = X s(t) + N_t$$

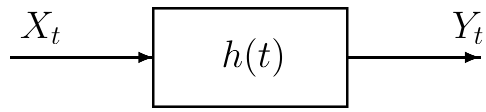
and the receiver needs to find the minimum mean square (MMSE) estimate for X based on Y_t, $0 \leq t \leq T$.

(a) By considering an appropriate KL expansion of Y_t with basis functions $\phi_1(t) = s(t)$ and $\phi_i(t)$ orthogonal to $\phi_1(t)$ for $i > 1$, show that all but one coefficients are orthogonal to X. Express this coefficient in terms of X, $s(t)$ and N_t.

(b) Using part (a) or otherwise, derive $\widehat{X}$, the MMSE estimate of X given Y_t for $0 \leq t \leq T$.

Problem 6 (4/60, equally weighted parts)

Let X_t be a zero-mean wide-sense stationary (WSS) random process with autocorrelation function $R_X(\tau) = e^{-|\tau|}$. Suppose that X_t is processed via a linear time-invariant (LTI) system as shown below.



(a) For this part, assume that $h(t) = e^{-bt}u(t)$ for some positive constant b. What is $S_Y(\omega)$, the power spectral density (PSD) of the output random process Y_t?

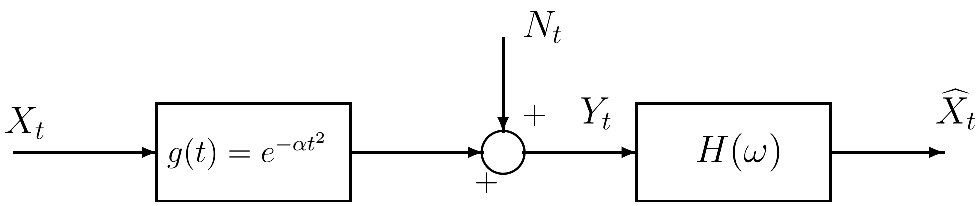
(b) For this part, assume that you have no direct information about the impulse response of the LTI system, but that you know the cross-correlation between the output and input

$$R_{YX}(\tau) = e^{-\tau}u(\tau) - 2e^{-2\tau}u(\tau) + e^{-3\tau}u(\tau) .$$

Find a possible impulse response $h(t)$. Is your answer unique? If so, explain why. If not, specify another possible $h(t)$.

Problem 7 (8/60, unequally weighted parts)

Consider the system shown below, where (i) X_t is a wide-sense stationary (WSS) random process with mean 0 and autocorrelation function $R_X(\tau) = e^{-\lambda|\tau|}$ with $\lambda > 0$; (ii) the WSS random process N_t is a white noise process that is independent of X_t, and has mean 0 and autocorrelation function $R_N(\tau) = \sigma^2\delta(\tau)$; (iii) the system that takes X_t as an input is linear time-invariant (LTI) with impulse response $g(t) = e^{-\alpha t^2}$ with $\alpha > 0$.



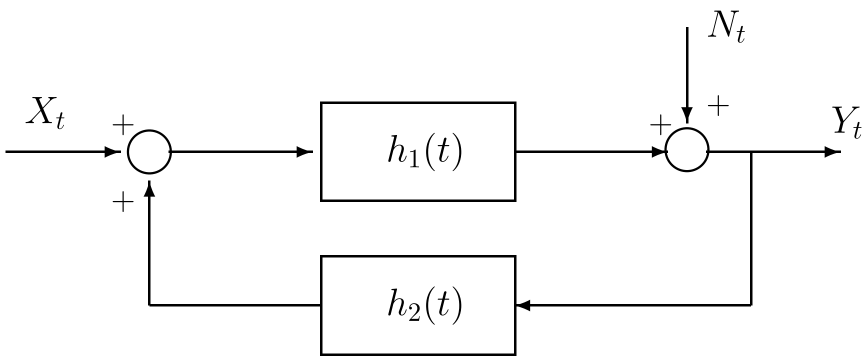
(a) (2 points) Find $S_X(\omega)$, the power spectral density (PSD) of the input random process X_t.

(b) (2 points) Find $S_Y(\omega)$, the power spectral density (PSD) of the output random process Y_t.

(c) (4 points) Find the frequency response $H(\omega)$ of the noncausal Wiener filter that produces an MMSE estimate of X_t based on the observation of Y_t, $-\infty < t < +\infty$.

Problem 8 (8/60, equally weighted parts)

Let X_t and N_t be *independent* zero-mean wide-sense stationary (WSS) random processes with autocorrelation functions $R_X(\tau) = e^{-3|\tau|}$ and $R_N(\tau) = \delta(\tau)$. Suppose that X_t is processed in the fashion shown below where both systems are linear time-invariant (LTI) with $h_1(t) = 2\delta(t)$ and $h_2(t) = 2e^{-5t}u(t)$.



(a) Find $S_Y(\omega)$, the power spectral density (PSD) at the output Y_t.

(b) Find the frequency response of the noncausal Wiener filter $H(\omega)$ that takes Y_t, $-\infty < t < +\infty$, as input and produces the minimum mean square error (MMSE) estimate for X_t given Y_t.

Bonus Problem (5/60)

Suppose that you randomly place two marks on a stick of length one. More specifically, one mark is placed at distance X from the left end of the stick and the other mark is placed at distance Y from the left end of the stick, with X and Y being i.i.d. random variables uniformly distributed in $[0, 1]$. What is the probability that if you cut the stick at both marks, you can form a triangle? [**Hint:** For a triangle with sides of lengths a, b and c we need $a + b > c$ *and* $a + c > b$ *and* $b + c > a$.]