Solutions to ECE 434 Final Exam, Spring 2003

Problem 1 *(21 points)* Indicate true or false for each statement below and *justify your answers*. (One third credit is assigned for correct true/false answer without correct justification.)
(a) If $\mathcal{H}(z)$ is a positive type z-transform, then so is $\cosh(\mathcal{H}(z))$. (Recall that $\cosh(x) = \frac{e^x+e^{-x}}{2}$.)
(b) If X is a m.s. differentiable stationary Gaussian random process, then for t fixed, X_t is independent of the derivative at time t: X'_t.
(c) If $X = (X_t : t \in \mathbb{R})$ is a WSS, m.s. differentiable, mean zero random process, then X is mean ergodic in the mean square sense.
(d) If $M = (M_t : t \geq 0)$ is a martingale with $E[M_t^2] < \infty$ for each t, then $E[M_t^2]$ is increasing in t (i.e. $E[M_s^2] \leq E[M_t^2]$ whenever $s \leq t$.)
(e) If $X_1, X_2, \ldots,$ is a sequence of independent, exponentially distributed random variables with mean one, then there is a finite constant K such that $P[X_1 + \cdots + X_n \geq 2n] \leq K\exp(-n^2)$ for all n.
(f) If X and Y are random variables such that $E[X^2] < \infty$ and $E[X|Y] = Y$, then $E[X^2|Y^2] = E[X|Y]^2$.
(g) If $N = (N_t : t \geq 0)$ is a random process with $E[N_t] = \lambda t$ and $E[N_s N_t] = \lambda \min\{s,t\}$ for $s, t \geq 0$, then N is a Poisson process.

(a) TRUE, The power series expansion of cosh(z) about zero is absolutely convergent yielding $\cosh(\mathcal{H}(z)) = 1 + \frac{(\mathcal{H}(z))^2}{2!} + \frac{(\mathcal{H}(z))^4}{4!} + \cdots$, and sums and products of positive type functions are positive type.
(b) TRUE, since $C_{X'X}(0) = C'_X(0) = 0$, and uncorrelated Gaussians are independent.
(c) FALSE, for a counter example let $X_t = U$ for all t, where U is a mean zero random variable with $0 < \mathrm{Var}(U) < \infty$.
(d) TRUE, since for $s < t$, $(M_t - M_s) \perp M_s$, so $E[M_t^2] = E[M_s^2] + E[(M_t - M_s)^2] \geq E[M_s^2]$.
(e) FALSE, since Cramèrs theorem implies that for any $\epsilon > 0$, $P[X_1 + \cdots + X_n \geq 2n] \geq \exp(-(l(2)+\epsilon)n)$ for all sufficiently large n.
(f) FALSE. For example let X have mean zero and positive variance, and let $Y \equiv 0$.
(g) FALSE. For example it could be that $N_t = W_t + \lambda t$, where W is a Wiener process with parameter $\sigma^2 = \lambda$.

Problem 2 *(12 points)* Let N be a Poisson random process with rate $\lambda > 0$ and let $Y_t = \int_0^t N_s ds$.
(a) Sketch a typical sample path of Y and find $E[Y_t]$.
(b) Is Y m.s. differentiable? Justify your answer.
(c) Is Y Markov ? Justify your answer.
(d) Is Y a martingale? Justify your answer.

(a) Your sketch should show that Y is continuous and piecewise linear. The slope of Y is 0 on the first interval, 1 on the second interval, 2 on the third interval, etc. $E[Y_t] = \int_0^t E[N_s]ds = \int_0^t \lambda s ds = \frac{\lambda t^2}{2}$.
(b) YES, since the integral of a m.s. continuous process is m.s. differentiable. $Y' = N$.
(c) N0, because knowing both Y_t and $Y_{t-\epsilon}$ for a small $\epsilon > 0$ determines the slope N_t with high probability, allowing a better prediction of $Y_{t+\epsilon}$ than knowing Y_t alone.
(d) NO, for example because a requirement for Y to be a martingale is $E[Y_t] = E[Y_0]$ for all t.

Problem 3 *(12 points)* Let $X_t = U\sqrt{2}\cos(2\pi t) + V\sqrt{2}\sin(2\pi t)$ for $0 \leq t \leq 1$, where U and V are independent, $N(0,1)$ random variables, and let $N = (N_\tau : 0 \leq \tau \leq 1)$ denote a real-valued Gaussian white noise process with $R_N(\tau) = \sigma^2\delta(\tau)$ for some $\sigma^2 \geq 0$. Suppose X and N are independent. Let $Y = (Y_t = X_t + N_t : 0 \leq t \leq 1)$. Think of X as a signal, N as noise, and Y as an observation.
(a) Describe the Karhunen-Loève expansion of X. In particular, identify the nonzero eigenvalue(s) and the corresponding eigenfunctions.
There is a complete orthonormal basis of functions $(\phi_n : n \geq 1)$ which includes the eigenfunctions found in part (a) (the particular choice is not important here), and the Karhunen-Loève expansions of N and Y can be given using such basis. Let $\tilde{N}_i = (N, \phi_i) = \int_0^1 N_t \phi_i(t)dt$ denote the i^{th}

coordinate of N. The coordinates $(\tilde{N}_1, \tilde{N}_2, \ldots)$ are $N(0, \sigma^2)$ random variables and $U, V, \tilde{N}_1, \tilde{N}_2, \ldots$ are independent. Consider the Karhunen-Loève expansion of Y, using the same orthonormal basis.
(b) Express the coordinates of Y in terms of $U, V, \tilde{N}_1, \tilde{N}_2, \ldots$ and identify the corresponding eigenvalues (i.e. the eigenvalues of $(R_Y(s,t) : 0 \leq s, t \leq 1)$).
(c) Describe the minimum mean square error estimator $\widehat{U}$ of U given $Y = (Y_t : 0 \leq t \leq 1)$, and find the minimum mean square error. Use the fact that observing Y is equivalent to observing the random coordinates appearing in the KL expansion of Y.
(a)Let $\phi_1(t) = \sqrt{2}\cos(2\pi t)$ and $\phi_2(t) = \sqrt{2}\sin(2\pi t)$. Then ϕ_1 and ϕ_2 are orthonormal and the K-L expansion of X is simply $X = U\phi_1(t) + V\phi_2(t)$. The nonzero eigenvalues are $\lambda_1 = \lambda_2 = 1$. We could write $X \leftrightarrow (U, V, 0, 0, \ldots)$. Remark: Since $\lambda_1 = \lambda_2$, any linear combination of these two eigenfunctions is also an eigenfunction. Any two orthonormal functions with the same linear span as ϕ_1 and ϕ_2 could be used in place of the two functions given. For example, another correct choice of eigenfunctions is $\xi_n(\omega) = \exp(2\pi jnt)$ for integers n. We also know this choice works because X is a WSS periodic random process. For this choice, ξ_1 and ξ_{-1} are the eigenfunctions with corresponding eigenvalues $\lambda_{-1} = \lambda_1 = 1$, and $\{\xi_1, \xi_{-1}\}$ and $\{\phi_1, \phi_2\}$ have the same linear span.
(b) $Y \leftrightarrow (U + \tilde{N}_1, V + \tilde{N}_2, \tilde{N}_3, \tilde{N}_4, \ldots)$ and the eigenvalues are $2, 2, 1, 1, \ldots$. (c) Observing $Y = (Y_t : 0 \leq t \leq 1)$ is equivalent to observing the coordinates of Y. Only the first coordinate of Y is relevant – the other coordinates of Y are independent of U and the first coordinate of Y. Thus, we need to estimate U given $\tilde{Y}_1$, where $\tilde{Y}_1 = (Y, \phi_1) = U + \tilde{N}_1$. The estimate is given by $\frac{\mathrm{Cov}(U,\tilde{Y}_1)}{\mathrm{Var}(\tilde{Y}_1)}\tilde{Y}_1 = \frac{1}{1+\sigma^2}\tilde{Y}_1$ and the covariance of error is $\frac{\mathrm{Var}(U)\mathrm{Var}(\tilde{N}_1)}{\mathrm{Var}(U)+\mathrm{Var}(\tilde{N}_1)} = \frac{\sigma^2}{1+\sigma^2}$.

Problem 4 *(7 points)* Let $(X_k : k \in \mathbb{Z})$ be a stationary discrete-time Markov process with state space $\{0, 1\}$ and one-step transition probability matrix $P = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$. Let $Y = (Y_t : t \in \mathbb{R})$ be defined by $Y_t = X_0 + (t \times X_1)$.
(a) Find the mean and covariance functions of Y.
(b) Find $P[Y_5 \geq 3]$.
(a) The distribution of X_k for any k is the probability vector π solving $\pi = P\pi$, or $\pi = (\frac{1}{2}, \frac{1}{2})$. Thus $P[(X_0, X_1) = (0,0)] = P[(X_0, X_1) = (1,1)] = \frac{1}{2}\frac{3}{4} = \frac{3}{8}$, and $P[(X_0, X_1) = (0,1)] = P[(X_0, X_1) = (1,0)] = \frac{1}{2}\frac{1}{4} = \frac{1}{8}$. Thus, $E[X_i] = E[X_i^2] = \frac{1}{2}$ and $E[X_0 X_1] = \frac{3}{8}$ so $\mathrm{Var}(X_i) = \frac{1}{4}$ and $\mathrm{Cov}(X_0, X_1) = \frac{3}{8} - (\frac{1}{2})^2 = \frac{1}{8}$. Thus, $E[Y_t] = \frac{1+t}{2}$ and $\mathrm{Cov}(Y_s, Y_t) = \mathrm{Cov}(X_0 + sX_1, X_0 + tX_1) = \mathrm{Var}(X_0) + (s+t)\mathrm{Cov}(X_0, X_1) + st\mathrm{Var}(X_1) = \frac{1}{4} + \frac{s+t}{8} + \frac{st}{4}$.
(b) Since $Y_5 = X_0 + 5X_1$, the event $\{Y_5 \geq 3\}$ is equal to the event $\{X_1 = 1\}$, which has probability 0.5.

Problem 5 *(6 points)* Let Z be a Gauss-Markov process with mean zero and autocorrelation function $R_Z(\tau) = e^{-|\tau|}$. Find $P[Z_2 \geq 1 + Z_1 | Z_1 = 2, Z_0 = 0]$.
Since Z is a Markov process, the conditional distribution of Z_2 given Z_0 and Z_1 depends only on Z_1. Note the if Z_2 is estimated by Z_1, then the minimum mean square error estimator is $E[Z_2|Z_1] = \frac{\mathrm{Cov}(Z_2,Z_1)Z_1}{\mathrm{Var}(Z_1)} = e^{-1}Z_1$, and the estimation error is independent of Z_1 and is Gaussian with mean zero and variance $\mathrm{Var}(Z_2) - \frac{\mathrm{Cov}(Z_2,Z_1)^2}{\mathrm{Var}(Z_1)} = 1 - e^{-2}$. Thus, given $Z_1 = 2$, the conditional distribution of Z_2 is Gaussian with mean $2e^{-1}$ and variance $1 - e^{-2}$. Thus, the desired conditional probability is the same as the probabilty a $N(2e^{-1}, 1 - e^{-2})$ random variable is greater than or equal to 3. This probability is $Q\left(\frac{3-2e^{-1}}{\sqrt{1-e^{-2}}}\right)$.

Problem 6 *(10 points)* Let X be a real-valued, mean zero stationary Gaussian process with $R_X(\tau) = e^{-|\tau|}$. Let $a > 0$. Suppose X_0 is estimated by $\widehat{X}_0 = c_1 X_{-a} + c_2 X_a$ where the constants c_1 and c_2 are chosen to minimize the mean square error (MSE).
(a) Use the orthogonality principle to find c_1, c_2, and the resulting minimum MSE, $E[(X_0 - \widehat{X}_0)^2]$. (Your answers should depend only on a.)
(b) Use the orthogonality principle again to show that $\widehat{X}_0$ as defined above is the minimum MSE estimator of X_0 given $(X_s : |s| \geq a)$. (This implies that X has a two-sided Markov property.) (a)The constants must be selected so that $X_0 - \widehat{X}_0 \perp X_a$ and $X_0 - \widehat{X}_0 \perp X_{-a}$., or equivalently $e^{-a} - [c_1 e^{-2a} + c_2] = 0$ and

$e^{-a} - [c_1 + c_2 e^{-2a}] = 0$. Solving for c_1 and c_2 (one could begin by subtracting the two equations) yields $c_1 = c_2 = c$ where $c = \frac{e^{-a}}{1+e^{-2a}} = \frac{1}{e^a+e^{-a}} = \frac{1}{2\cosh(a)}$.

The corresponding minimum MSE is given by $E[X_0^2] - E[\widehat{X}_0^2] = 1 - c^2 E[(X_{-a}+X_a)^2] = 1 - c^2(2+2e^{-2a}) = \frac{1-e^{-2a}}{1+e^{2a}} =$ Tanh(a).

(b) The claim is true if $(X_0 - \widehat{X}_0) \perp X_u$ whenever $|u| \geq a$.

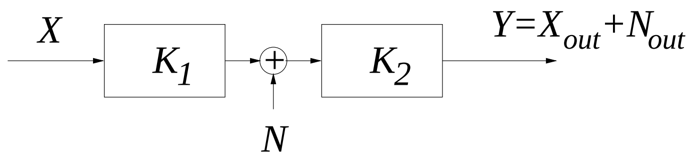
If $u \geq a$ then $E[(X_0 - c(X_{-a} + X_a))X_u] = e^{-u} - \frac{1}{e^a+e^{-a}}(e^{-a-u} + e^{a-u}) = 0$.

Similarly if $u \leq -a$ then $E[(X_0 - c(X_{-a} + X_a))X_u] = e^{u} - \frac{1}{e^a+e^{-a}}(e^{a+u} + e^{-a+u}) = 0$.

The orthogonality condition is thus true whenever $|u| \geq a$, as required.

Problem 7 *(12 points)*

Suppose X and N are jointly WSS, mean zero, continuous time random processes with $R_{XN} \equiv 0$. The processes are the inputs to a system with the block diagram shown, for some transfer functions $K_1(\omega)$ and $K_2(\omega)$:



Suppose that for every value of ω, $K_i(\omega) \neq 0$ for $i = 1$ and $i = 2$. Because the two subsystems are linear, we can view the output process Y as the sum of two processes, X_{out}, due to the input X, plus N_{out}, due to the input N. Your answers to the first four parts should be expressed in terms of K_1, K_2, and the power spectral densities S_X and S_N.

(a) What is the power spectral density S_Y?

(b) What is the signal-to-noise ratio at the output (equal to the power of X_{out} divided by the power of N_{out})?

(c) Suppose Y is passed into a linear system with transfer function H, designed so that the output at time t is $\widehat{X}_t$, the best (not necessarily causal) linear estimator of X_t given $(Y_s : s \in \mathbb{R})$. Find H.

(d) Find the resulting minimum mean square error.

(e) The correct answer to part (d) (the minimum MSE) does not depend on the filter K_2. Why?

(a) $S_Y = |K_1K_2|^2 S_X + |K_2|^2 S_N$, where for notational brevity we suppress the argument (ω) for each function.

(b) $SNR_{output} = \frac{\int_{-\infty}^{\infty} |K_1K_2|^2 S_X \frac{d\omega}{2\pi}}{\int_{-\infty}^{\infty} |K_2|^2 S_N \frac{d\omega}{2\pi}}$.

(c) $H = \frac{S_{XY}}{S_Y} = \frac{\overline{K_1K_2} S_X}{|K_1K_2|^2 S_X + |K_2|^2 S_N}$.

(d)

$$\begin{aligned} MMSE &= \int_{-\infty}^{\infty} S_X - S_Y|H|^2 \frac{d\omega}{2\pi} \\ &= \int_{-\infty}^{\infty} \frac{|K_2|^2 S_X S_N}{|K_1|^2|K_2|^2 S_X + |K_2|^2 S_N} \frac{d\omega}{2\pi} \\ &= \int_{-\infty}^{\infty} \frac{S_X S_N}{|K_1|^2 S_X + S_N} \frac{d\omega}{2\pi} \end{aligned}$$

(e) Since K_2 is invertible, for the purposes of linear estimation of X, using the process Y is the same as using the process Y filtered using a system with transfer function $\frac{1}{K_2(\omega)}$. Equivalently, the estimate of X would be the same if the filter K_2 were dropped from the original system.