Homework Set 6 - Solutions

ECE 530

1

1.1 Show that $\hat{x}$ is actually the least-squares solution to the system

$$\begin{bmatrix} A \\ I \end{bmatrix} x = \begin{bmatrix} b \\ c \end{bmatrix}$$

$$\text{where } \hat{x} = \arg\min_x ||Ax - b||_2^2 + ||x - c||_2^2$$

By the normal equations, the least-squares solution is

$$x = \left(\begin{bmatrix} A \\ I \end{bmatrix}^T \begin{bmatrix} A \\ I \end{bmatrix} \right)^{-1} \begin{bmatrix} A \\ I \end{bmatrix}^T \begin{bmatrix} b \\ c \end{bmatrix} = (A^T A + I)^{-1}(A^T b + c)$$

$$\begin{aligned}
\text{Given } ||Ax - b||_2^2 + ||x - c||_2^2 &= (Ax - b)^T (Ax - b) + (x - c)^T (x - c) \\
&= x^T A^T Ax - 2x^T A^T b + b^T b - 2x^T c + c^T c \\
&= x^T (A^T A + I)x - 2x^T (A^T b + c) + b^T b + c^T c \\
\nabla_x \left(||Ax - b||_2^2 + ||x - c||_2^2 \right) &= 2(A^T A + I)x - 2(A^T b + c) = 0 \\
\implies \hat{x} &= (A^T A + I)^{-1}(A^T b + c) = x \quad \checkmark
\end{aligned}$$

1.2 Is this solution unique?

For $Ax = b$, the solution is unique when A is full column rank. The solution is always unique because $\begin{bmatrix} A \\ I \end{bmatrix}$ is always full column rank.

2 DC Power Flow State Estimation

```
syms th2 th3

sigma = [0.1, 0.1, 0.1]; % assumed error std. dev.
R = diag(sigma.^2);

Z = [3.2; 2.5; 0.8];    % measurements
fsym = [10*sin(-th2);
        10*sin(-th3);
        10*sin(th2-th3)];
Hsym = [-10*cos(-th2),      0;
        0,                  -10*cos(-th3);
        10*cos(th2-th3),    -10*cos(th2-th3)];

 toSub = [th2; th3];
 daSub = [0;0];         % Initial Start
 f = subs(fsym,toSub,daSub);
 H = subs(Hsym,toSub,daSub);

HRH = inv(transpose(H)*inv(R)*H);
x(:,1) = double(daSub+HRH*transpose(H)*inv(R)*(Z-f));
fprintf('Delta2 is %f \nDelta3 is %f\n',x(1,1),x(2,1))
```

```
Delta2 is -0.270000
Delta3 is -0.300000
```

Note: While the problem gives P_{12} = 320 MW, P_{13} = 250 MW, and P_{23} = 80 MW, by the picture, P_{23} flows $from$ 3 to 2, meaning that by the convention for P_{12} and P_{13}, P_{23} should be negative. Using that measurement in Z gives the solution

```
Delta2 is -0.323333
Delta3 is -0.246667
```

Because of the ambiguity, both answers were accepted.

3 Givens Rotations

Given $\begin{bmatrix} 3 & 2 \\ 2 & -1 \\ 6 & 1 \end{bmatrix}$

Step 1: zero out A[3,1]	$a = 2$	$b = 6$
Step 2: zero out A[2,1]	$a = 3$	$b = -6.3245$
Step 3: zero out A[3,2]	$a = 1.5360$	$b = -1.2649$

$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.3162 & 0.9487 \\ 0 & -0.9487 & -0.3162 \end{bmatrix} \qquad G_1^T A = \begin{bmatrix} 3 & 2 \\ -6.3245 & -0.6325 \\ 0 & -1.2649 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} 0.4286 & 0.9035 & 0 \\ -0.9035 & 0.4286 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad G_2^T A = \begin{bmatrix} 7 & 1.4286 \\ 0 & 1.5360 \\ 0 & -1.2649 \end{bmatrix}$$

$$G_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7719 & 0.6357 \\ 0 & -0.6357 & 0.7719 \end{bmatrix} \qquad G_3^T A = \begin{bmatrix} 7 & 1.4286 \\ 0 & 1.9898 \\ 0 & 0 \end{bmatrix}$$

$$Q = G_1 G_2 G_3 = \begin{bmatrix} 0.4286 & 0.6974 & 0.5744 \\ 0.2857 & -0.7077 & 0.6462 \\ 0.8571 & -0.1128 & -0.5026 \end{bmatrix} \qquad R = U = \begin{bmatrix} 7 & 1.4286 \\ 0 & 1.9898 \\ 0 & 0 \end{bmatrix}$$

4 Key Issues of the Normal Equations

1. While A may be sparse, $A^T A$ is much less sparse and consequently requires more computing and storage resources for the solution.
2. $A^T A$ may be numerically less well-conditioned than A.

5 SVD DECOMPOSITION

If $A = USV^T$, then U and V are square unitary matrices $\implies \begin{cases} UU^T = U^TU = I \\ VV^T = V^TV = I \end{cases}$

Furthermore, S is a diagonal matrix and the property of being a diagonal matrix is preserved over multiplication.

Then,

$$\begin{aligned} AA^T &= (USV^T)(USV^T)^T = USV^T(V^T)^TS^TU^T = USV^TVS^TU^T \\ &= USS^TU^T \end{aligned}$$

The diagonalizable form of a matrix is given by XDX^{-1}, where X is nonsingular and D is diagonal. The matrix SS^T is diagonal (preservation of diagonality) and because U is square and unitary it is also invertible with $U^T = U^{-1}$.

Similarly,

$$\begin{aligned} A^TA &= (USV^T)^T(USV^T) = (V^T)^TS^TU^TUSV^T \\ &= VS^TSV^T \end{aligned}$$

And $D = S^TS$ and $X = V$.