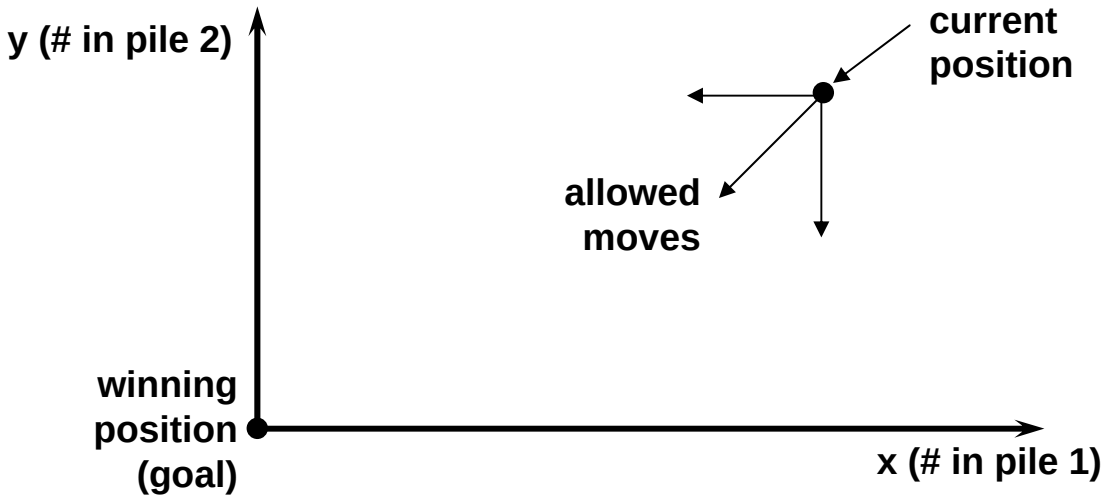
# A Sequential Task Specification

- an example of how hard it is to find a problem with no parallelism
  - taken from…that's right!...ECE190
  - another ECE190 MP3 (Fall 2008), with some simplifications

- a game for two players (Wythoff's game)
  - two piles of sticks
  - goal: take the last stick(s)
  - allowed moves
    - take any # > 0 from either pile
    - take the same # > 0 from both piles




- note
  - no ties, no move cycles, finite # steps to goal (bounded by x + y)
  - thus all positions are forced win/forced loss if played correctly
- question
  - which positions are forced win/loss?
  - (lots of wins; we'll focus on specifying forced loss positions)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- start with the goal state, (0,0)
  - if (x,y) = (0,0) on your turn, you lose
  - thus it's a forced lose

| | |
|---|---|
| 0 | L |
| | 0 |

- first set of forced wins
  - any position from which you can reach the goal state in one move
  - for any P > 0
    - (P,0) or
    - (P,P) or
    - (0,P)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6 | W | | | | | | W |
| 5 | W | | ? | | | W | |
| 4 | W | | | | W | | |
| 3 | W | | | W | | | |
| 2 | W | | W | | | ? | |
| 1 | W | W | | | | | |
| 0 | L | W | W | W | W | W | W |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |


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- any position
  - for which all possible moves are forced wins
  - is a forced loss
  - thus (1,2) and (2,1) are forced losses

- Which induces more forced wins: for any P > 1,
  - (P+1,1) or
  - (P,2) or
  - (P+1,P) or
  - (P,P+1) or
  - (2,P) or
  - (1,P+1)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6 | W | W | W | ? | | W | W |
| 5 | W | W | W | | W | W | W |
| 4 | W | W | W | W | W | W | |
| 3 | W | W | W | W | W | | ? |
| 2 | W | L | W | W | W | W | W |
| 1 | W | W | L | W | W | W | W |
| 0 | L | W | W | W | W | W | W |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |

- the next forced loss is (3,5) and (5,3), and so forth

- but let's use induction…


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- starting point
  - for simplicity, write forced loss cases as (x,y) with x < y
  - observe that any forced loss (x,y) differs by some amount; call it k=y-x
    - base case (0,0) has k=0
    - next couple of cases, (1,2) and (3,5), have k=1, k=2
  - forced loss case for any k is unique
    - assume two cases for some k: (x,x+k) and (z,z+k)
    - assume x<z without loss of generality
    - but (z,z+k) can move to (x,x+k) in one move
      (take z-x from both piles)
    - contradiction: (z,z+k) is not a forced loss, but a forced win!
  - however, forced loss case may not exist for all k

- now use algorithmic induction
  - base case is (0,0); we know it works and thus k=0 case exists
  - assume sequence (x0,y0), (x1,y1), …, (x(k-1), y(k-1)) to some k
  - find method
    - to determine xk and
    - show that (xk, xk + k) is a forced loss

- solution: let xk be
  - the smallest whole number
  - that does not appear in any previous x or y value
  - note: good luck parallelizing the search!

- proof
  - three possible move types from (xk, xk + k)
  - we'll consider one at a time
  - show that all result in forced wins


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- first two moves: reduce xk OR reduce both
  - for some $p > 0$
    - $(xk, xk + k)$ moves to $(xk – p, xk + k)$ OR
    - $(xk, xk + k)$ moves to $(xk – p, xk + k – p)$
  - by choice of xk, $xk – p$ appears in a previous forced loss
  - thus, for some i, either $xi = xk – p$ or $xi + i = xk – p$
  - first case
    - forced loss at $(xk – p, xk – p + i)$
    - note that $i < k$, so $xk – p + i < xk + k – p < xk + k$
    - and both $(xk – p, xk + k – p)$ and $(xk – p, xk + k)$ are forced wins
  - second case
    - forced loss at $(xk – p – i, xk – p)$
    - reverse the indices: clearly $xk – p – i < xk + k – p < xk + k$
    - again, both $(xk – p, xk + k – p)$ and $(xk – p, xk + k)$ are forced wins


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- last move: reduce xk + k
  - for some p > 0, (xk, xk + k) moves to (xk, xk + k - p)
  - first case: |k-p| < k
    - difference k – p is now covered by some previous forced loss case
    - but xk does not appear (by choice of xk) in that case
    - thus result is forced win (forced loss for any k is unique)
  - second case: |k-p| ≥ k (which means p ≥ 2k)
    - reverse indices to put smaller value first: (xk + k – p, xk)
    - xk + k – p < xk, thus xk + k – p appears in a previous forced loss
    - for some i, either xi = xk + k – p or xi + i = xk + k – p
    - first case
      - forced loss at (xk + k – p, xk + k – p + i)
      - again, since i < k, xk + k – p + i < xk
      - and (xk + k – p, xk) is a forced win
    - second case
      - forced loss at (xk + k – p – i, xk + k – p)
      - reverse the indices: clearly xk + k – p – i < xk
      - again, (xk + k – p, xk) is a forced win

- and…
  - there is a closed-form solution!
  - (didn't see one last time I looked, but it's old…maybe as old as 1907?)
  - however, I suggest that you try to find it yourself
  - since without it you can't parallelize…
  - [darn]


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