Problem 1: State true or false with a 1 line justification [24 points]

a. The DFT spectrum of a signal is symmetric around $F_s/2$, where $F_s$ is the sampling frequency.
   False. Only if the signal is real.

b. GPS location estimation also synchronizes the clock of the mobile device with the satellite clock.
   True. Since location estimation depends on time of flight, location and clock need to be jointly solved.

c. Signal strength is a good indicator of distance in indoor environments.
   False. Absorption due to obstacles, and constructive/destructive interference through multipath cause signal strength to vary erratically in indoor environments.

d. In RADAR, more the number of access points (APs), better will be the accuracy.
   True. Higher number of APs increases the dimensionality. That will help improve accuracy.

e. For a static object, estimating gravity well can determine the orientation of the object.
   False. Object can rotate on the plane perpendicular to the gravity vector. One vector is therefore insufficient.

f. If a transmitter is in the near field, then the location of the object can be determined from a receiver array.
   False. There exists a locus of points which will have the same phase difference.

g. If phase can be measured precisely, then it is adequate to measure the distance between a transmitter and a receiver.
   False. Phase wraps and hence is not adequate for distance measurements.

h. In free fall, an accelerometer will record 0 on all three axes.
   True. There is no reaction force acting on the accelerometer.

i. Since white noise has a zero mean, we can keep tracking an object’s location by integrating IMU data over long period of time.
   False. Both, a stationary object and one with a constant velocity will have a 0 acceleration. Impossible to know the state of the object (moving/stationary) by observing windows of time. Therefore, integrating will always incur localization error.

j. Order of rotations does not matter. That is: $R_x(R_y(R_z)) \equiv R_y(R_z(R_x))$.
   False. Rotation matrix are not commutative. Example: Think of a dice with opposite faces: 1-6, 2-5, 3-4. Suppose 1 is on top and 2 facing you. Now spin dice to have 3
facing you, and then turn the dice to have the 3 on top. Now reverse this order; First
make 2 face top and then spin it. 2 is still on top.

k. A model based localization approach based on RSS (such as that explored by second part of RADAR)
will work well in outer space.
True. RSS model is exact in space since there are no corruptions from multipath and
obstacles.

l. Localization approach used in UnLoc will fail in an open area since it will have no remarkable
landmarks to reset.
True. UnLoc is based on landmarks and a lack of landmarks will mean UnLoc has no
reset points.

Problem 2: Explain or argue briefly [21 points]

a. Explain (5 points): Why is clock synchronization between GPS satellites so crucial to localize mobile
devices on the ground.
Because otherwise you would need to model clock error between each satellite and the
receiver. In that case, the number of unknowns is always greater than equations.

b. Argue (5 points): There is some order in which the UnLoc landmarks get estimated (i.e., the ordering
is not random).
First, the seed landmarks are estimated and then the organic landmarks are estimated.
But within organic landmarks, the landmarks closest to seed landmarks get estimated
first followed by landmarks at iteratively increasing distances.

c. Argue (5 points): KNN is not always better than NN localization. If so, give an example, if not,
argue why not.
KNN is not always better. If the user revisits a location that was part of the collected
signatures, KNN will still estimate the location based on the average of K neighbors.
This will reduce KNN’s accuracy.

d. Argue both for and against (6 points): In a 2-antenna AoA setup, Alice wants to separate the anten-
nas by $2\lambda$ distance. What are the benefits and drawbacks over a separation of $\lambda/2$?
Disadvantage: The $\lambda/2$ distance between antennas allows spanning the entire 180 de-
grees. So there is no ambiguity. Advantage: A $2\lambda$ distance improves the resolution. So
if the approximate direction is known, it is beneficial to use a larger separation between
antennas.

Problem 3: Math nuggets and derivations [20 points]

a. Derive the Nyquist sampling theorem from first principles.
See notes.

b. Derive the steering matrix for AoA estimation from first principles.
Suppose the gap between $n$ antennas is $d$ each. Signals incident at an angle of $\theta$ would
create a phase difference $\phi = d \cdot \cos \theta$. Now, the transmitted signal $x(t)$ when received at
the first antenna would be: $y_1(t) = \cos(2\pi ft)$. At the next antenna, the phase difference
\( \phi \) will manifest as: \( y_2(t) = \cos(2\pi ft + \phi) \). Continuing this trend, the received signal will be:

\[
\begin{bmatrix}
y_1(t) \\
y_2(t) \\
y_3(t) \\
\vdots \\
y_n(t)
\end{bmatrix} =
\begin{bmatrix}
\cos(2\pi ft) \\
\cos(2\pi ft + \phi) \\
\cos(2\pi ft + 2\phi) \\
\vdots \\
\cos(2\pi ft + (n-1)\phi)
\end{bmatrix} =
\begin{bmatrix}
e_0 \\
e^\phi \\
e^{2\phi} \\
\vdots \\
e^{(n-1)\phi}
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t) \\
\vdots \\
x_m(t)
\end{bmatrix}
\]

For \( m \) such directions,

\[
\begin{bmatrix}
y_1(t) \\
y_2(t) \\
y_3(t) \\
\vdots \\
y_n(t)
\end{bmatrix} =
\begin{bmatrix}
e_0 \\
e^\phi \\
e^{2\phi} \\
\vdots \\
e^{(n-1)\phi}
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t) \\
\vdots \\
x_m(t)
\end{bmatrix}
\]

\[\begin{align*}
P(s_k | m_{1:n}) &= \\
\text{See notes}
\end{align*}\]

d. Derive the chain rule for the joint probability, \( P(A, B, C, D) \)

\begin{align*}
P(A, B, C, D) &= P(A|B, C, D) \cdot P(B, C, D) \\
P(B, C, D) &= P(B|C, D) \cdot P(C, D) \\
P(C, D) &= P(C|D) \cdot P(D)
\end{align*}

Substituting:

\[
\therefore P(A, B, C, D) = P(A|B, C, D) \cdot P(B|C, D) \cdot P(C|D) \cdot P(D)
\]

Problem 4: 3D Orientation [15 points]

Assume that the magnitude of acceleration due to gravity is \( g \) and the magnitude of the earth’s magnetic field is \( m \). Also assume that we are at the equator. For a mobile phone assume x-axis is parallel to the phone’s width (shorter side), y-axis is parallel to the phone’s length (longer side), z-axis penetrates through the phone’s glass surface.

a. A static mobile phone is supported on a table and shows accelerometer reading: \([0, 1, 0] g\). Does this represent the complete orientation of the phone? If yes, what is the orientation matrix. If no, give a reason why not.

No. The orientation is ambiguous if only one vector is known. Rotation around that vector will give different orientations.

b. The above phone is showing a magnetometer reading of \([1, 0, 0] m\). Is this possible? What is the orientation matrix for the phone?

Yes, this is a valid configuration. The orientation matrix is:

\[
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

c. A static mobile phone is reporting the following measurements on its accelerometer: \([-0.3789, 0.2775, 0.8828] g\). It is reporting the following magnetometer readings: \([0.5829, 0.8125, -0.0053] m\). What is
its orientation matrix?
The multiplication factors for the acceleration due to gravity (g) and the earth’s magnetic have been factored out for us. This makes things simpler. Now suppose the rotation matrix is \( R \). Then, we have to orient the device such that it is aligned with the gravity and the magnetic north.

\[
R^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.3789 \\ 0.2775 \\ 0.8828 \end{bmatrix}
\]

\[
R^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5829 \\ 0.8125 \\ -0.0053 \end{bmatrix}
\]

This implies that the second and third row of the matrix \( R \) is completely known to us. We just have the first row to calculate. Now,

\[
R \begin{bmatrix} 0.5829 & -0.3789 \\ 0.8125 & 0.2775 \\ -0.0053 & 0.8828 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

Expanding \( R \),

\[
\begin{bmatrix} a_1 & a_2 & a_3 \\ 0.5829 & 0.8125 & -0.0053 \\ -0.3789 & 0.2775 & 0.8828 \end{bmatrix} \begin{bmatrix} 0.5829 & -0.3789 \\ 0.8125 & 0.2775 \\ -0.0053 & 0.8828 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

This yields two equations for \( a_1, a_2, a_3 \):

\[
0.5829a_1 + 0.8125a_2 - 0.0053a_3 = 0
\]

\[
-0.3789a_1 + 0.2775a_2 + 0.8828a_3 = 0
\]

Since \( R \) is a rotation matrix, it must be orthogonal. Hence, \( |R| = 1 \). This gives us the third equation:

\[
a_1(0.8125*0.8828+0.0053*0.2775) - a_2(0.5829*0.8828-0.0053*0.3789) + a_3(0.5829*0.2775+0.8125*0.3789) = 1
\]

\[
\therefore 0.7187a_1 - 0.5126a_2 + 0.4696a_3 = 1
\]

Solving the three equations for \( a_1, a_2, \) and \( a_3 \), we obtain \( R \) as:

\[
R = \begin{bmatrix} 0.7188 & -0.5127 & 0.4697 \\ 0.5829 & 0.8125 & -0.0053 \\ -0.3789 & 0.2775 & 0.8828 \end{bmatrix}
\]

Another possible approach is to perform cross product of the two given vectors to obtain the third.
Problem 5: Programming module: Beamforming and AoA [20 points]

- Code up a small simulator which has a device X with N antennas (uniformly separated by distance DIS).
- Device X is receiving a signal from different angles THETA.
- The signal is at frequency FREQ = 100kHz and is being transmitted by a far field transmitter Y.
- Now, implement the basic AoA sensing algorithm.

a. Assuming you don’t know THETA, plot the AoA spectrum for DIS = wavelength, half wavelength, and quarter wavelength (note that c = f * lambda ).

b. Add increasing random noise to the received signal, and re-plot the AoA spectrum but this time for DIS = half wavelength only. What do you observe as you increase noise?

c. Now add a reflection signal coming from a different angle BETA and re-plot the AoA spectrum only for DIS = half wavelength. Can you still detect the AoA accurately?

d. Keep adding reflection signals and determine at which point the AoA becomes inaccurate.