Note: Problems (or parts of problems) marked with a star (★) are required for graduate students to receive 4 credit hours; undergraduate students who solve these problems will receive extra credit points.

Submission: Write your name, netid, and U for undergrad/G for grad in the upper right-hand corner of the first page of your written solutions. Typewritten solutions will receive 5 extra credit points.

Problems to be handed in

1 Consider the RC circuit shown in Figure 1 of the lecture on the Johnson–Nyquist noise. Suppose that the noisy voltage source E is bandlimited, i.e., its power spectral density has the form

$$S_E(\omega) = \begin{cases} 2kTR, & |\omega| \le \omega_0 \\ 0, & \text{otherwise} \end{cases},$$

where ω_0 is the bandwidth in rad/s.

(a) Compute the average power $\mathbf{E}[V_t^2]$ of the voltage across the capacitor.

(b) Let β denote the ratio of the noise bandwidth ω_0 to the RC circuit bandwidth $1/RC$, i.e., $\beta = \omega_0 RC$. How does the expression from part (a) behave in the limit $\beta \to \infty$?

2 In the lecture on $1/f$ noise, we encountered a continous-time stochastic signal $X = (X_t)_{t \in \mathbb{R}}$ taking values in $\mathsf{X} = \{-1, +1\}$, with the probabilities $p_t(-) := \mathbf{P}[X_t = -1]$ and $p_t(+) := \mathbf{P}[X_t = +1]$ obeying the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} p_t(-) & p_t(+) \end{pmatrix} = \begin{pmatrix} p_t(-) & p_t(+) \end{pmatrix} \begin{pmatrix} -\alpha & \alpha \\ \alpha & -\alpha \end{pmatrix}.$$

By diagonalizing the matrix on the right, show that, for any $t \in \mathbb{R}$ and any $\tau \ge 0$, the solution is given by

$$\begin{pmatrix} p_{t+\tau}(-) & p_{t+\tau}(+) \end{pmatrix} = \begin{pmatrix} p_t(-) & p_t(+) \end{pmatrix} \begin{pmatrix} \frac{1+e^{-2\alpha\tau}}{2} & \frac{1-e^{-2\alpha\tau}}{2} \\ \frac{1-e^{-2\alpha\tau}}{2} & \frac{1+e^{-2\alpha\tau}}{2} \end{pmatrix}.$$

3 In the lecture on $1/f$ noise, we have considered the probability distribution of the relaxation time $T_0 = ce^{\Delta E/kT}$, where c is a positive constant, k is the Boltzmann constant, T is the ambient temperature in Kelvin, and ΔE is a random energy gap distributed uniformly on the interval $[\Delta_0, \Delta_1]$.

(a) Prove that the pdf of T_0 is given by

$$g(t_0) = \begin{cases} \frac{kT}{(\Delta_1 - \Delta_0)t_0}, & ce^{\Delta_0/kT} \le t_0 \le ce^{\Delta_1/kT} \\ 0, & \text{otherwise} \end{cases}.$$

(b) Compute the mean and the variance of T_0.

4 (★) Let $N = (N_t)_{t\geq 0}$ be a Poisson process with rate λ, and $T = (T_k)_{k\in\mathbb{Z}_+}$ be the arrival times of N (with $T_0 = 0$). Let M be a given $n \times n$ Markov matrix. Consider a continuous-time stochastic signal $X = (X_t)_{t\geq 0}$ with finite state space $\mathsf{X} = \{0, \dots, n-1\}$ that evolves as follows: it starts from $X_0 = 0$ and stays the same until the next arrival, at which point it changes randomly to a different state with probabilities prescribed by M. That is, $X_t = 0$ for $t < T_1$; then at $t = T_1$, $X_t = y$ with probability $M(X_0, y) = M(0, y)$, for each $y \in \mathsf{X}$. Then the state X_t stays the same until $t = T_2$, at which point it changes randomly to a new state y' with probability $M(X_{T_1}, y')$, etc.

(a) Prove that X is a Markov process.

(b) Let p_t denote the probability distribution of X_t, i.e., $p_t(x) = \mathbf{P}[X_t = x]$ for each $x \in \mathsf{X}$. Prove the following explicit formula for p_t:

$$p_t = p_0 e^{\lambda t(M - I_n)},$$

where I_n is the $n \times n$ identity matrix, p_0 is the initial state distribution (in this case, $p_0(x) = 1$ if $x = 0$ and 0 otherwise), and the matrix exponential e^A for a square matrix A is defined as

$$e^A \triangleq \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Hint: Use the fact that the number of state transitions between times 0 and t is equal to N_t, the number of arrivals by time t, then apply the law of total probability.

(c) Consider the binary case $\mathsf{X} = \{0, 1\}$ with $M(0,0) = M(1,1) = \frac{1}{2}$. Compute the matrix $e^{\lambda t(M - I_2)}$ explicitly. What can you say about the long-term behavior of p_t – i.e., will it converge to a limiting distribution, and, if the answer is "yes," how fast is the convergence?

(d) How does this relate to the continuous-time Markov process we have used to model conductance fluctuations in our analysis of $1/f$ noise?