ECE 498MR Homework 4 (due 4/12/17, beginning of class)

Note: Problems (or parts of problems) marked with a star (∗) are required for graduate students to receive 4 credit hours; undergraduate students who solve these problems will receive extra credit points.

Submission: Write your name, netid, and u for undergrad/g for grad in the upper right-hand corner of the first page of your written solutions. Typewritten solutions will receive 5 extra credit points.

Problems to be handed in

1 Consider the RC circuit shown in Figure 1 of the lecture on the Johnson–Nyquist noise. Suppose that the noisy voltage source $E$ is bandlimited, i.e., its power spectral density has the form

$$S_E(\omega) = \begin{cases} 2kTR, & |\omega| \leq \omega_0 \\ 0, & \text{otherwise} \end{cases},$$

where $\omega_0$ is the bandwidth in rad/s.

(a) Compute the average power $E[V^2_t]$ of the voltage across the capacitor.

(b) Let $\beta$ denote the ratio of the noise bandwidth $\omega_0$ to the RC circuit bandwidth $1/RC$, i.e., $\beta = \frac{\omega_0}{RC}$. How does the expression from part (a) behave in the limit $\beta \to \infty$?

2 In the lecture on $1/f$ noise, we encountered a continuous-time stochastic signal $X = (X_t)_{t \in \mathbb{R}}$ taking values in $X = \{-1, +1\}$, with the probabilities $p_t(-) := P[X_t = -1]$ and $p_t(+) := P[X_t = +1]$ obeying the differential equation

$$\frac{d}{dt} \begin{pmatrix} p_t(-) \\ p_t(+) \end{pmatrix} = \begin{pmatrix} \alpha & -\alpha \\ \alpha & -\alpha \end{pmatrix} \begin{pmatrix} p_t(-) \\ p_t(+) \end{pmatrix}.$$

By diagonalizing the matrix on the right, show that, for any $t \in \mathbb{R}$ and any $\tau \geq 0$, the solution is given by

$$\begin{pmatrix} p_{t+\tau}(-) \\ p_{t+\tau}(+) \end{pmatrix} = \begin{pmatrix} p_t(-) \\ p_t(+) \end{pmatrix} \begin{pmatrix} 1 + e^{-2\alpha\tau} & 1 - e^{-2\alpha\tau} \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

3 In the lecture on $1/f$ noise, we have considered the probability distribution of the relaxation time $T_0 = ce^{\Delta E/kT}$, where $c$ is a positive constant, $k$ is the Boltzmann constant, $T$ is the ambient temperature in Kelvin, and $\Delta E$ is a random energy gap distributed uniformly on the interval $[\Delta_0, \Delta_1]$.

(a) Prove that the pdf of $T_0$ is given by

$$g(t_0) = \begin{cases} \frac{kT}{(\Delta_1 - \Delta_0)t_0}, & ce^{\Delta_0/kT} \leq t_0 \leq ce^{\Delta_1/kT} \\ 0, & \text{otherwise} \end{cases}.$$

(b) Compute the mean and the variance of $T_0$. 
4 (⋆) Let $N = (N_t)_{t \geq 0}$ be a Poisson process with rate $\lambda$, and $T = (T_k)_{k \in \mathbb{Z}_+}$ be the arrival times of $N$ (with $T_0 = 0$). Let $M$ be a given $n \times n$ Markov matrix. Consider a continuous-time stochastic signal $X = (X_t)_{t \geq 0}$ with finite state space $X = \{0, \ldots, n-1\}$ that evolves as follows: it starts from $X_0 = 0$ and stays the same until the next arrival, at which point it changes randomly to a different state with probabilities prescribed by $M$. That is, $X_t = 0$ for $t < T_1$; then at $t = T_1$, $X_t = y$ with probability $M(X_0, y) = M(0, y)$, for each $y \in X$. Then the state $X_t$ stays the same until $t = T_2$, at which point it changes randomly to a new state $y'$ with probability $M(X_{T_1}, y')$, etc.

(a) Prove that $X$ is a Markov process.

(b) Let $p_t$ denote the probability distribution of $X_t$, i.e., $p_t(x) = P[X_t = x]$ for each $x \in X$. Prove the following explicit formula for $p_t$:

$$p_t = p_0 e^{\lambda t (M - I_n)},$$

where $I_n$ is the $n \times n$ identity matrix, $p_0$ is the initial state distribution (in this case, $p_0(x) = 1$ if $x = 0$ and 0 otherwise), and the matrix exponential $e^A$ for a square matrix $A$ is defined as

$$e^A \triangleq \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

*Hint:* Use the fact that the number of state transitions between times 0 and $t$ is equal to $N_t$, the number of arrivals by time $t$, then apply the law of total probability.

(c) Consider the binary case $X = \{0, 1\}$ with $M(0, 0) = M(1, 1) = \frac{1}{2}$. Compute the matrix $e^{\lambda t (M - I_2)}$ explicitly. What can you say about the long-term behavior of $p_t$ – i.e., will it converge to a limiting distribution, and, if the answer is “yes,” how fast is the convergence?

(d) How does this relate to the continuous-time Markov process we have used to model conductance fluctuations in our analysis of $1/f$ noise?