4 Stochastic signal processing

Now that we have at least some idea of what stochastic signals are, we can begin to analyze the behavior of systems with stochastic inputs. Let’s recall that a system is anything that transforms an input signal into an output signal. Let’s keep things simple and work with continuous-time input and output signals, and consider the case of deterministic inputs first. A system $S$ transforms an input signal $x: \mathbb{R} \rightarrow \mathbb{R}$ into an output signal $y: \mathbb{R} \rightarrow \mathbb{R}$, and we denote this fact by writing $y = S[x]$. This notation signifies the fact that the input to $S$ is the entire signal $x$, and the output of $S$ is the entire signal $y$. Thus, the value $y(t) = S[x](t)$ of the output at time $t$ may, in principle, depend on all $x(s), s \in \mathbb{R}$.

Clearly, at this level of abstraction there is not a whole lot that can be done. So, it is useful to single out various types of systems:

- **causal** — when the current value of the output is not affected by the future values of the input. Formally, for any $t \in \mathbb{R}$ and for any two inputs $x_1$ and $x_2$, such that $x_1(s) = x_2(s)$ for all $s \leq t$, we have
  $$S[x_1](t) = S[x_2](t).$$

- **memoryless** — when the current value of the output depends only on the current value of the input. Formally, for any $t \in \mathbb{R}$ and for any two inputs $x_1$ and $x_2$, such that $x_1(t) = x_2(t)$, we have
  $$S[x_1](t) = S[x_2](t).$$

  Any memoryless system is causal.

- **time-invariant** — when the output due to a time-shifted version of the input is the time-shifted version of the output. Formally, given a signal $v: \mathbb{R} \rightarrow \mathbb{R}$ and an arbitrary $\tau \in \mathbb{R}$, define its time shift $v_\tau: \mathbb{R} \rightarrow \mathbb{R}$ by $v_\tau(t) \triangleq v(t - \tau)$. Then
  $$S[x_\tau](t) = (S[x])_t.$$

- **linear** — when the output due to a superposition of inputs is the superposition of the outputs. Formally, given any two input signals $x_1, x_2 : \mathbb{R} \rightarrow \mathbb{R}$ and any two real coefficients $\alpha_1, \alpha_2$,
  $$S[\alpha_1 x_1 + \alpha_2 x_2] = \alpha_1 S[x_1] + \alpha_2 S[x_2].$$

We have already seen how Markov chains can be viewed as outputs of deterministic systems driven by stochastic inputs. We will come back to this description a bit later, but first we will examine the scenario where a continuous-time stochastic signal is used as an input to a linear system, and describe the properties of the resulting output signal in terms of the properties of the input and of the system.
4.1 Linear time-invariant systems: a quick review

In ECE 210, we mostly deal with systems that are linear and time-invariant (LTI), because for such systems the relationship between the input and the output is particularly easy to write down. Each linear system is described by a function $h : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, such that, for any input signal $x$, the output is given by the superposition integral

$$S[x](t) = \int_{-\infty}^{\infty} h(t, \tau)x(\tau)d\tau$$

(as a memory refresher, prove that Eq. (4.1) specifies a linear system). The function $h$ is called the impulse response of the system because it is explicitly given by

$$h(t, \tau) = S[x](t) \quad \text{when } x(t) = \delta(t-\tau).$$

When the system is also time-invariant, $h(t, \tau) = h(t+t_0, \tau+t_0)$ for any $t_0 \in \mathbb{R}$, so, overloading the notation a bit, we can rewrite (4.1) as a convolution:

$$S[x](t) = h \ast x(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau.$$  

While computing convolutions is an important character-building part of ECE 210, things are much easier when we pass to the frequency domain using Fourier transforms. The Fourier transform of a function $g : \mathbb{R} \rightarrow \mathbb{R}$ (whenever it exists) is defined as

$$\hat{g}(\omega) \triangleq \int_{-\infty}^{\infty} g(t)e^{-i\omega t} dt,$$

where the argument $\omega \in \mathbb{R}$ is the frequency\(^1\) and $i = \sqrt{-1}$ is the imaginary unit. To get back to the time domain, we use the Fourier inversion formula

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\omega)e^{i\omega t} d\omega.$$  

The Fourier transform $\hat{h}$ of the impulse response $h$ is called the transfer function of the system, and the time-domain expression $y = h \ast x$ becomes

$$\hat{y} = H\hat{x}$$

in the frequency domain, where $\hat{x}$ is the Fourier transform of the input $x$, and $\hat{y}$ is the Fourier transform of the output $y$.

Remark on notation. In ECE 210, you were probably used to writing uppercase letters for Fourier transforms, like this: $X(\omega)$ instead of $\hat{x}(\omega)$. However, this will cause confusion when we start

\(^1\)Not to be confused with the generic element of the sample space $\Omega$ — hopefully, the meaning will be clear from the context.
dealing with stochastic inputs, since in that case \( X \) will already denote the time-domain input. I will still write \( H \) instead of \( \hat{H} \) because one must pay at least some respect to tradition, and because we will hardly ever use \( H \) to denote stochastic signals.

So, our next order of business is to adopt this formalism to the case when the input is a stochastic signal. But first we need to discuss the concept of stationarity.

### 4.2 Stationarity: weak and strong

As we have seen, both the Wiener process and the Poisson process have *stationary increments*. For example, if \( W = (W_t)_{t \geq 0} \) is a Wiener process, then, for any two times \( 0 \leq s \leq t \) and for any \( r \geq -s \), the distribution of the increment \( W_t - W_s \) is the same as the distribution of the increment \( W_{t+r} - W_{s+r} \), namely Gaussian with mean 0 and variance \( D(t-s) \). Thus, the statistical properties of the increments of \( W \) are unaffected by time shifts.

This property turns out to be rather useful, so we abstract it into a definition: A stochastic signal \( X = (X_t)_{t \in T} \) is *(strongly)* stationary if, for any \( n \in \mathbb{N} \), any finite sequence of times \( t_1, t_2, \ldots, t_n \in T \), and any \( r \in \mathbb{R} \), such that \( t_1 + r, \ldots, t_n + r \in T \),

\[
(X_{t_1}, X_{t_2}, \ldots, X_{t_n}) \overset{d}{=} (X_{t_1+r}, X_{t_2+r}, \ldots, X_{t_n+r}) \tag{4.5}
\]

(the notation \( U \overset{d}{=} V \) means that the random quantities \( U \) and \( V \) have the same distribution). When \( X \) has a discrete state space \( X \), the stationarity condition (4.5) means that

\[
\Pr[X_{t_1} = x_1, X_{t_2} = x_2, \ldots, X_{t_n} = x_n] = \Pr[X_{t_1+r} = x_1, X_{t_2+r} = x_2, \ldots, X_{t_n+r} = x_n]
\]

for all \( x_1, \ldots, x_n \in X \); when \( X \) has a continuous state space \( X \), (4.5) means that

\[
\Pr[a_1 \leq X_{t_1} \leq b_1, \ldots, a_n \leq X_{t_n} \leq b_n] = \Pr[a_1 \leq X_{t_1+r} \leq b_1, \ldots, a_n \leq X_{t_n+r} \leq b_n]
\]

for all subintervals \([a_1, b_1], \ldots, [a_n, b_n]\) of \( X \). This definition covers discrete and continuous time. For example, any i.i.d. process \( X = (X_k)_{k \in \mathbb{Z}_+} \) is strongly stationary.

However, for many purposes, strong stationarity is too much to ask for. Instead, we consider the following weaker notion: Let \( X = (X_t)_{t \in T} \) be a stochastic signal with a continuous state space \( X \). Then we say that \( X \) is *weakly stationary* (and write WS, for short) if it has the following two properties:

1. The mean function \( m_X(t) = \mathbb{E}[X_t] \) is constant as a function of \( t \):

\[
m_X(t) = \mu, \quad \forall t \in T. \tag{4.6}
\]

2. For any two times \( s, t \in T \) and any \( r \in \mathbb{R} \) such that \( s + r \in T \) and \( t + r \in T \),

\[
R_X(s, t) = R_X(s + r, t + r). \tag{4.7}
\]

Here, \( R_X(s, t) = \mathbb{E}[X_s X_t] \) is the autocorrelation function of \( X \).
This property is much weaker than strong stationarity: one can easily construct examples of stochastic signals that are very nonstationary in the sense of (4.5), yet are weakly stationary. (Of course, any strongly stationary process with a continuous state space is also weakly stationary.) Now let us examine some implications of weak stationarity. While the definition applies to any $T$, we will focus on the easy case when $T$ is closed under addition and subtraction: if $s, t \in T$, then $s + t \in T$ and $s - t \in T$. Then $0 \in T$. In that case, from (4.6) we get $\mu = E[X_0]$, and from (4.7) with $r = -t$ we get

$$E[X^2_t] = R_X(t, t) = R_X(0, 0) = E[X_0^2].$$

Thus, $E[X^2_t] = \sigma^2$ for some $\sigma \geq 0$. Now, if we use (4.7) with $r = -s$ and $r = -t$, we have

$$R_X(s, t) = R_X(0, t - s) \quad \text{and} \quad R_X(t, s) = R_X(0, s - t).$$

Since $R_X(s, t) = R_X(t, s)$, we conclude that $R_X(s, t)$ depends only on $\tau = t - s$. Thus, for a weakly stationary stochastic process $X = (X_t)_{t \in T}$, we can overload the notation and write its autocorrelation function as $R_X(\tau)$. This really means that

$$R_X(\tau) = E[X_tX_{t+\tau}] = E[X_tX_{t-\tau}], \quad \forall t, \tau \in T$$

and, in particular, implies that $R_X(\tau) = R_X(-\tau)$.

Before moving on, let us look at an example. Let $A, B$ be two jointly distributed real-valued random variables, and consider the following stochastic signal $X = (X_t)_{t \in \mathbb{R}}$:

$$X_t = A \cos \omega t + B \sin \omega t \quad (4.8)$$

(here, $\omega$ is a deterministic angular frequency, not to be confused with a generic element of some probability space $(\Omega, \mathcal{F}, \mathbb{P})$). This is an example of a deterministic signal with stochastic parameters. What are the conditions on $A$ and $B$ for this signal to be weakly stationary? We claim that $X$ is weakly stationary if and only if the following three conditions are satisfied:

1. $E[A] = E[B] = 0$ (both $A$ and $B$ have zero mean).
2. $\text{Var}[A] = \text{Var}[B] = \sigma^2$ ($A$ and $B$ have the same variance).
3. $E[AB] = 0$ ($A$ and $B$ are uncorrelated).

We will only prove the statement that if $X$ is WS, then $A$ and $B$ have to satisfy the above conditions; the converse will be a homework problem. So, suppose that $X$ is WS. Then

$$m_X(t) = E[X_t] = E[A \cos \omega t + B \sin \omega t] = E[A] \cos \omega t + E[B] \sin \omega t,$$

and the only way for $m_X(t)$ to be a constant is to have $E[A] = E[B] = 0$, because otherwise it will depend on $t$. This proves Item 1. Now, again by the WS assumption, $E[X^2_0] = R_X(0)$ must be a constant. Since $X_0 = A$ and $X_{\pi/2\omega} = B$, we have $E[X^2_0] = E[A^2] = \text{Var}[A]$ and $E[X^2_{\pi/2\omega}] = E[B^2] = \sigma^2$. Therefore, $\sigma^2 = E[X^2_0] = E[X^2_{\pi/2\omega}] = \sigma^2$ for some $\sigma \geq 0$. This proves Item 2. Now, we can show that $E[AB] = 0$. Since $X_0 = A$ and $X_{\pi/2\omega} = B$, we have $E[AB] = E[A^2] = \text{Var}[A]$ and $E[B^2] = E[A^2] = \sigma^2$. Therefore, $E[AB] = 0$. This proves Item 3.
Var\[B\] (we have used the fact that both \(A\) and \(B\) have zero mean). Thus, \(\text{Var}[A] = \text{Var}[B] = \sigma^2\) for some \(\sigma \geq 0\). This proves Item 2. Finally, for any \(t, \tau \in T\) we write

\[
R_X(t, t + \tau) = \mathbb{E}[X_tX_{t+\tau}]
\]

\[
= \mathbb{E}[(A \cos \omega t + B \sin \omega t)(A \cos \omega (t + \tau) + B \sin \omega (t + \tau))]
\]

\[
= \mathbb{E}[A^2] \cos \omega t \cos \omega (t + \tau) + \mathbb{E}[B^2] \sin \omega t \sin \omega (t + \tau)
+ \mathbb{E}[AB] (\cos \omega t \sin \omega (t + \tau) + \sin \omega t \cos \omega (t + \tau))
\]

\[
= \sigma^2 \cos \omega \tau + \mathbb{E}[AB] \sin \omega (2t + \tau),
\]

where we have used trigonometric identities and the fact that \(\mathbb{E}[A^2] = \mathbb{E}[B^2] = \sigma^2\). We see that, unless \(\mathbb{E}[AB] = 0\), \(R_X(t, t + \tau)\) will depend on both \(t\) and \(\tau\), which would violate the assumption that \(X\) is WS. This proves Item 3.

As we will learn next, weak stationarity is preserved by linear time invariant (LTI) systems: if \(X = (X_t)_{t \in \mathbb{R}}\) is a WS input to an LTI system, then the output \(Y = (Y_t)_{t \in \mathbb{R}}\) is also WS, and its mean and correlation functions can be explicitly computed from those of \(X\) and from the impulse response of the system.

### 4.3 Systems with stochastic inputs: the LTI case

Consider an LTI system with impulse response \(h\) and fix a stochastic signal \(X = (X_t)_{t \in \mathbb{R}}\). Then the output signal \(Y = (Y_t)_{t \in \mathbb{R}}\) is related to \(X\) via the convolution integral, as in (4.2):

\[
Y_t = \int_{-\infty}^{\infty} h(t - \tau)X_\tau d\tau.
\]  

(4.9)

Of course, this is purely formal, since each \(X_\tau\) is a random variable, and it takes some care to endow an integral like (4.9) with rigorous meaning. We will happily ignore all this fuss and just assume that the above integral is well-defined. Nevertheless, it is still a random quantity, and may not admit a closed-form expression. However, as we will now see, the output mean \(m_Y(t)\), the input-output crosscorrelation \(R_{XY}(s, t)\), and the output autocorrelation \(R_Y(s, t)\) can be expressed in terms of the input mean \(m_X(t)\), the input autocorrelation \(R_X(s, t)\), and the impulse response \(h\).

First, let’s do the mean:

\[
m_Y(t) = \mathbb{E}[Y_t] = \mathbb{E} \left[ \int_{-\infty}^{\infty} h(t - \tau)X_\tau d\tau \right].
\]  

(4.10)

In general, interchanging expectations and integrals is a delicate matter, but we will be cavalier about it and just do it:

\[
\mathbb{E} \left[ \int_{-\infty}^{\infty} h(t - \tau)X_\tau d\tau \right] = \int_{-\infty}^{\infty} \mathbb{E}[h(t - \tau)X_\tau]d\tau = \int_{-\infty}^{\infty} h(t - \tau)\mathbb{E}[X_\tau]d\tau.
\]

Now, \(\mathbb{E}[X_\tau] = m_X(\tau)\), so, putting everything together, we arrive at the pleasing formula

\[
m_Y(t) = \int_{-\infty}^{\infty} h(t - \tau)m_X(\tau)d\tau \equiv h \ast m_X(t).
\]  

(4.11)
That is, the output mean is given by the convolution of the input mean with the impulse response! We immediately note that if \( m_X(t) \) is constant, i.e., \( m_X(t) = \mu_X \) for all \( t \), then \( m_Y(t) \) is also constant and equal to

\[
\mu_Y = \mu_X \int_{-\infty}^{\infty} h(t) dt.
\]

Here, we assume that the integral of \( h \) over the entire real line exists and is finite. We can also express this in terms of the transfer function \( H \) by noting that

\[
\int_{-\infty}^{\infty} h(t) dt = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt \bigg|_{\omega = 0} = H(0),
\]

so we arrive at the following formula: if \( m_X(t) = \mu_X \) for all \( t \), then \( m_Y(t) = H(0) \mu_X \).

Encouraged by our success, let’s compute the input-output crosscorrelation next:

\[
R_{XY}(s, t) = \mathbb{E}[X_s Y_t] \\
= \mathbb{E} \left[ \int_{-\infty}^{\infty} h(t-\tau) X_s X_\tau d\tau \right] \\
= \int_{-\infty}^{\infty} h(t-\tau) \mathbb{E}[X_s X_\tau] d\tau \\
= \int_{-\infty}^{\infty} h(t-\tau) R_X(s, \tau) d\tau.
\]

This almost looks like a convolution, except that \( R_X \) is a function of two arguments. Note, however, that we are integrating over the second argument of \( R_X \). So, we can write

\[
R_{XY}(s, t) = h \ast_2 R_X(s, t),
\]

where the subscript 2 on the asterisk indicates that we convolve only over the second argument of the input autocorrelation \( R_X \):

\[
h \ast_2 R_X(s, t) \triangleq \int_{-\infty}^{\infty} h(t-\tau) R_X(s, \tau) d\tau.
\]

This simplifies considerably when \( X \) is weakly stationary: in that case, \( R_X(s, t) = R_X(t-s) \), and so

\[
R_{XY}(s, t) = h \ast_2 R_X(s, t) \\
= \int_{-\infty}^{\infty} h(t-\tau) R_X(s, \tau) d\tau \\
= \int_{-\infty}^{\infty} h(t-\tau) R_X(\tau-s) d\tau \\
= h \ast R_X(t-s)
\]
is just the ordinary convolution. Now we note that $R_{XY}(s, t)$ depends only on $t - s$, so we can write $R_{XY}(s, t) = R_{XY}(t - s)$. Consequently, we arrive at the formula

$$R_{XY}(\tau) = h \ast R_X(\tau), \quad (4.12)$$

where $R_{XY}(\tau) = \mathbb{E}[X_t Y_{t+\tau}]$ for all $t \in \mathbb{R}$. Thus, when the input $X$ is weakly stationary, the input-output crosscorrelation is given by the convolution of the impulse response and the input autocorrelation!

Finally, let’s look at the output autocorrelation. Again, blithely interchanging the order of expectation and integration, we have

$$R_Y(s, t) = \mathbb{E}[Y_s Y_t]$$

$$= \mathbb{E} \left[ \int_{-\infty}^{\infty} h(s - \tau)X_{\tau}d\tau \int_{-\infty}^{\infty} h(t - \tau')X_{\tau'}d\tau' \right]$$

$$= \mathbb{E} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(s - \tau)h(t - \tau')X_{\tau}X_{\tau'}d\tau d\tau' \right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(s - \tau)h(t - \tau') \mathbb{E}[X_{\tau}X_{\tau'}]d\tau d\tau'$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(s - \tau)h(t - \tau') R_X(\tau, \tau')d\tau d\tau'.$$

If we perform the integration over $\tau'$ first, we can recognize the $\ast_2$ operation:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(s - \tau)h(t - \tau') R_X(\tau, \tau')d\tau d\tau' = \int_{-\infty}^{\infty} h(s - \tau) \left( \int_{-\infty}^{\infty} h(t - \tau') R_X(\tau, \tau')d\tau' \right) d\tau$$

$$= h \ast_2 R_X(\tau, t)$$

$$= \int_{-\infty}^{\infty} h(s - \tau) R_{XY}(\tau, t)d\tau.$$

So, if we now define the “partial convolution” $\ast_1$ over the first argument, we can write

$$R_Y(s, t) = h \ast_1 R_{XY}(s, t) = h \ast_1 (h \ast_2 R_X)(s, t).$$

This deceptively simple-looking formula hides a lot of complexity. But, once again, things simplify if $X$ is weakly stationary. In that case, $h \ast_2 R_X(s, t) = h \ast R_X(t - s) = R_{XY}(t - s)$, and therefore

$$h \ast_1 (h \ast_2 R_X)(s, t) = \int_{-\infty}^{\infty} h(s - \tau) R_{XY}(\tau, t)d\tau$$

$$= \int_{-\infty}^{\infty} h(s - \tau) R_{XY}(t - \tau)d\tau. \quad (4.13)$$

Again, this integral looks very much like a convolution, but here we hit a snag. In general, a convolution integral of two functions $f$ and $g$ will look like this:

$$\int_{-\infty}^{\infty} f(t - \tau) g(\tau - s)d\tau,$$
i.e., if we add the arguments of $f$ and $g$ in the integrand, the integration variable $\tau$ will cancel, and it is not hard to show that the integral is equal to $f \ast g(t - s)$. But if we add up the arguments of $h$ and $R_{XY}$ in the integrand of (4.13), we get $(s - \tau) + (t - \tau) = s + t - 2\tau$, and $\tau$ most certainly does not cancel! Fortunately, there is a nice hack out of this conundrum: given $h$, define another function $\tilde{h}$ by setting $\tilde{h}(t) \triangleq h(-t)$. For obvious reasons, we call $\tilde{h}$ the time reversal of $h$. Then, with this definition, we can rewrite (4.13) as

$$\int_{-\infty}^{\infty} h(s - \tau)R_{XY}(t - \tau)\,d\tau = \int_{-\infty}^{\infty} \tilde{h}(\tau - s)R_{XY}(t - \tau)\,d\tau,$$

and behold: $(\tau - s) + (t - \tau) = t - s$, and so we can write $h \ast_1 (h \ast_2 R_X)(s, t) = \tilde{h} \ast h \ast R_X(t - s)$. Thus, when the input $X$ is weakly stationary, the output autocorrelation is given by the double convolution:

$$R_Y(\tau) = \tilde{h} \ast h \ast R_X(\tau), \quad (4.14)$$

i.e., if the input $X$ is weakly stationary with autocorrelation $R_X(\tau)$, then the output $Y$ is also weakly stationary. Moreover, in that case the crosscorrelation $R_{XY}(s, t)$ also depends only on $t - s$, so we say that the input and the output are jointly weakly stationary: each of them is WS, and $R_{XY}(s, t) = R_{XY}(t - s)$ [although be careful: $R_{XY}(t - s) = E[X_t Y_s] \neq E[X_t Y_s] = R_{XY}(s - t)$].

### 4.4 Input-output relations in the frequency domain: power spectral densities

No one likes to compute convolution integrals (even though it builds character), so we pass to the frequency domain. Let $X$ be a WS stochastic signal with autocorrelation $R_X(\tau)$. The power spectral density of $X$, denoted by $S_X$, is simply the Fourier transform of $R_X$:

$$S_X(\omega) \triangleq \hat{R}_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)e^{-i\omega \tau}\,d\tau. \quad (4.15)$$

The autocorrelation function can be recovered from the power spectral density via the Fourier inversion formula (4.4):

$$R_X(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega)e^{i\omega \tau}\,d\omega. \quad (4.16)$$

The word “spectral” evokes frequency content, so this makes sense. But what do “power” and “density” mean? If we think about $X_t$ as a (random) current passing through a unit resistance, then the voltage across the resistor is also equal to $X_t$, by Ohm’s law. Consequently, the dissipated power at time $t$ is equal to

$$(\text{voltage at time } t) \cdot (\text{current at time } t) = X_t^2.$$

Thus, we may think of $E[X_t^2]$ as the average power dissipated at time $t$. Now, if $X$ is WS, then $E[X_t^2] = E[X_t X_t] = R_X(0)$. On the other hand, if we substitute $\tau = 0$ into the Fourier inversion formula (4.16), we get

$$E[X_t^2] = R_X(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega)d\omega. \quad (4.17)$$
Similarly, given two jointly WS stochastic signals $X$ and $Y$, we define their \textit{cross-power spectral density} $S_{XY}$ as the Fourier transform of the crosscorrelation function $R_{XY}$:

$$S_{XY}(\omega) = \hat{R}_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{i\omega \tau} d\tau.$$

Using power spectral densities, we can obtain frequency-domain forms of the results of the preceding section. For example, taking the Fourier transform of both sides of (4.12), we get

$$S_{XY}(\omega) = H(\omega)S_X(\omega). \quad (4.18)$$

Similarly, taking the Fourier transform of both sides of (4.14), we get

$$S_Y(\omega) = \tilde{H}(\omega)H(\omega)S_X(\omega), \quad (4.19)$$

where $\tilde{H}$ is the Fourier transform of the time-reversed impulse response $\tilde{h}$. However, we can express $\tilde{H}$ in terms of $H$:

$$\tilde{H}(\omega) = \int_{-\infty}^{\infty} \tilde{h}(\tau) e^{-i\omega \tau} d\tau$$

$$= \int_{-\infty}^{\infty} h(-\tau) e^{-i\omega \tau} d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau) e^{i\omega \tau} d\tau$$

$$= H(-\omega),$$

where in the penultimate line we have made the change of variable $\tau \to -\tau$. Now, since $h$ is real-valued, we see that $\tilde{H}(\omega) = H(-\omega)$ is just the complex conjugate of $H(\omega)$: $\tilde{H}(\omega) = H^*(\omega)$. Therefore, (4.19) simplifies to

$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega). \quad (4.20)$$

Thus, when a WS stochastic signal is used as an input to an LTI system, the effect in the frequency domain is to reshape the power spectral density in proportion to $|H|^2$.

\textbf{Example: white and colored noise.} Consider a zero-mean WS stochastic signal $Z$ with the flat power spectral density $S_Z(\omega) = q$, where $q > 0$ is some fixed constant. By analogy with white visible light that contains all spectral components, we call such a signal \textit{white noise} with intensity $q$. The autocorrelation function is then given by $R_Z(\tau) = q\delta(\tau)$. Since $Z$ is zero-mean,

$$R_Z(\tau) = C_Z(\tau) = E[Z_t Z_{t+\tau}]$$

for any $t, \tau \in \mathbb{R}$. In this case, $R_Z(\tau) = 0$ for all $\tau > 0$, i.e., $Z_t$ and $Z_{t'}$ are uncorrelated for all $t \neq t'$. Although white noise is a useful mathematical construction (for example, many stochastic signals
of interest can be interpreted as outputs of deterministic systems driven by white-noise inputs), it is unphysical. To see why, let us substitute $S_Z(\omega) = q$ into (4.17) to get

$$E[Z_t^2] = \frac{1}{2\pi} \int_{-\infty}^{\infty} q \, d\omega = +\infty.$$  

That is, pure white noise has infinite average power! Of course, we can simply continue using it while remaining mindful of its unphysical nature (just like we do with the unit impulse), or we can develop various approximations. For example, it is often possible to consider stochastic signals whose power spectra are flat and nonzero only in some band of frequencies. Thus, consider a WS stochastic signal $Z$ with

$$S_Z(\omega) = \begin{cases} q, & -\omega_0 \leq \omega \leq \omega_0 \\ 0, & \text{otherwise} \end{cases},$$  

which is finite. We can also compute the autocorrelation function of $Z$:

$$R_Z(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Z(\omega) e^{i\omega \tau} \, d\omega$$

$$= \frac{q}{2\pi} \int_{-\omega_0}^{\omega_0} e^{i\omega \tau} \, d\omega$$

$$= \frac{q}{2\pi} \cdot \frac{1}{i\tau} [e^{i\omega_0 \tau} - e^{-i\omega_0 \tau}]$$

$$= \frac{q}{\pi \tau} \cdot \frac{\sin \omega_0 \tau}{2i}$$

Recalling the definition of the sinc function\(^2\)

$$\text{sinc}(u) = \begin{cases} \frac{\sin \pi u}{\pi u}, & u \neq 0 \\ 1, & u = 0 \end{cases},$$

we can write

$$R_Z(\tau) = 2q f_0 \text{sinc}(2f_0 \tau),$$

where $f_0 = \omega_0 / 2\pi$ is the cutoff frequency in Hz. A plot of $R_Z$ with $q = 5$ and $\omega_0 = 4$ rad/s is shown in Fig. 1. Note that because $|R_Z(\tau)| > 0$ for all $\tau$, there are correlations between $Z_t$ and $Z_{t+\tau}$, but they decay to zero as $|\tau| \to \infty$.

\(^2\)This is the engineer’s sinc function. The mathematician’s sinc function is $\text{sinc}(u) = \frac{\sin u}{u}$. 

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The term “colored noise” refers to stochastic signals whose power spectra are roughly shaped like the corresponding colors of the visible spectrum. For example, the term “pink noise” refers to power spectra of the form \( S(\omega) \propto 1/\omega \). Later on, we will return to this noise model when discussing flicker noise in electronic devices.

4.4.1 Properties of power spectral densities

The power spectral density \( S_X \) of any WS stochastic signal \( X = (X_t)_{t \in \mathbb{R}} \) is real (i.e., \( S_X(\omega) \in \mathbb{R} \) for all \( \omega \)), even (i.e., \( S_X(-\omega) = S_X(\omega) \) for all \( \omega \)), and nonnegative (i.e., \( S_X(\omega) \geq 0 \) for all \( \omega \)).

Indeed, recall that \( S_X \) is the Fourier transform of the autocorrelation function \( R_X \). The autocorrelation function \( R_X \) takes real values, and it is also even:

\[
R_X(-\tau) = \mathbb{E}[X_t X_{t-\tau}] = \mathbb{E}[X_{t+\tau} X_t] = R_X(\tau).
\]

Therefore, using Euler’s formula, we have

\[
S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-i\omega \tau} d\tau \\
= \int_{-\infty}^{\infty} R_X(\tau) \cos \omega \tau d\tau + i \int_{-\infty}^{\infty} R_X(\tau) \sin \omega \tau d\tau.
\]

Since \( R_X(\tau) \) is even and \( \sin \omega \tau \) is odd, the second integral is identically zero, and we obtain the formula

\[
S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) \cos \omega \tau d\tau. \tag{4.23}
\]
Since the integrand takes real values, $S_X(\omega) \in \mathbb{R}$ for all $\omega$. Moreover, the integrand is even as a function of $\omega$, so $S_X$ is also an even function. This also implies that $R_X$ is given by

$$R_X(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) e^{i\omega \tau} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) \cos \omega \tau d\omega + \frac{i}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) \sin \omega \tau d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) \cos \omega \tau d\omega,$$

where we have used the same reasoning as before: $S_X(\omega) \sin \omega \tau$ is an odd function of $\omega$, so the second integral is zero.

It remains to show that $S_X(\omega) \geq 0$ for all $\omega$. Let $X$ be the input to a bandpass filter with the transfer function

$$H(\omega) = \begin{cases} 1, & \omega_0 \leq |\omega| \leq \omega_1 \\ 0, & \text{otherwise} \end{cases},$$

where $\omega_0 \leq \omega_1$ are arbitrary nonnegative constants. Let $Y$ denote the output stochastic signal. Then

$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega) = \begin{cases} S_X(\omega), & \omega_0 \leq |\omega| \leq \omega_1 \\ 0, & \text{otherwise} \end{cases}.$$

Now, using the Fourier inversion formula and the evenness of $S_X$,

$$R_Y(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_1}^{-\omega_0} S_X(\omega) d\omega + \frac{1}{2\pi} \int_{\omega_0}^{\omega_1} S_X(\omega) d\omega$$

$$= \frac{1}{\pi} \int_{\omega_0}^{\omega_1} S_X(\omega) d\omega.$$ 

Since $R_Y(0) = \mathbb{E}[Y^2_t] \geq 0$, we see that

$$\int_{\omega_0}^{\omega_1} S_X(\omega) d\omega \geq 0, \quad \forall \omega_0, \omega_1. \quad (4.24)$$

The only way for (4.24) to hold is if $S_X(\omega) \geq 0$ for all $\omega$.

In fact, for any function $S$ of frequency $\omega$ which is real, even, nonnegative, and satisfies the condition

$$a \overset{\triangle}{=} \int_{-\infty}^{\infty} S(\omega) d\omega < \infty,$$

we can construct a WS stochastic signal $X$, such that $S = S_X$. To prove this, consider the following stochastic signal:

$$X_t = \cos(\Omega t + \Theta),$$
where $\Omega$ and $\Theta$ are two mutually independent random variables. Let us assume that $a = \pi$ (otherwise, we can simply rescale $S$). We will take $\Theta \sim \text{Uniform}(0,2\pi)$, and we will choose the pdf $f_{\Omega}$ of $\Omega$ later. First, let us prove that $X$ is WS. For that, we have to compute its mean and autocorrelation functions. Since $\Omega$ and $\Theta$ are independent, we can write

$$m_X(t) = \mathbb{E}[X_t]$$

$$= \mathbb{E}[\cos(\Omega t + \Theta)]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f_{\Omega}(\omega) \cos(\omega t + \theta) d\theta d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f_{\Omega}(\omega) \left( \int_{0}^{2\pi} \cos(\omega t + \theta) d\theta \right) d\omega.$$ 

The integral in parentheses is equal to zero, so $m_X(t) = 0$ for all $t$. For the autocorrelation function, using the trigonometric identity $\cos u \cos v = \frac{1}{2} [\cos(u + v) + \cos(u - v)]$, we have

$$R_X(t, t + \tau) = \mathbb{E}[X_t X_{t+\tau}]$$

$$= \mathbb{E}[\cos(\Omega t + \Theta) \cos(\Omega(t + \tau) + \Theta)]$$

$$= \frac{1}{2} \mathbb{E}[\cos(2\Omega t + \Omega \tau + \Theta) + \cos(\Omega \tau)]$$

$$= \frac{1}{2} \mathbb{E}[\cos(2\Omega t + \Omega \tau + \Theta)] + \frac{1}{2} \mathbb{E}[\cos \Omega \tau].$$

Using the same reasoning as in the derivation of $m_X$, we see that the first expectation is identically zero, and therefore

$$R_X(t, t + \tau) = \frac{1}{2} \mathbb{E}[\cos \Omega \tau].$$ (4.25)

Since $m_X = 0$ and $R_X(t, t + \tau)$ depends only on $\tau$, $X$ is indeed WS. Now we will pick $f_{\Omega}$ to guarantee that $S_X = S$. To that end, we will use the Fourier inversion formula

$$R_X(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) \cos(\omega \tau) d\omega$$

(recall that $S_X$ is even). On the other hand, from (4.25) we have

$$R_X(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} f_{\Omega}(\omega) \cos(\omega \tau) d\omega.$$ 

In particular,

$$R_X(0) = \frac{1}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega$$

Let us take $f_{\Omega}(\omega) = \frac{S(\omega)}{\pi}$. This function is nonnegative and integrates to 1. Therefore, it is a valid pdf. By uniqueness of Fourier transforms, we have ensured that $S = \hat{R}_X$, and therefore $S_X = S$. 

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