Instructions

- You may not use any calculators, cell phones, earphones, or any other forms of electronics on this quiz.
- Show all your work to receive full credit for your answers.
- When you are asked to “calculate”, “determine”, or “find”, this means providing closed-form expressions (i.e., without summation or integration signs).
- Neatness counts. If we are unable to read your work, we cannot grade it.
- Turn in your entire booklet once you are finished. No extra booklet or papers will be considered.
- The asterisk mark (*) denotes convolution.

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**Problem 1 (30 points)**

Consider the causal, linear shift-invariant (LSI) system (with zero initial conditions) described in the following diagram:

(a) (5 points) Determine the linear constant coefficient difference equation relating the input $x[n]$ to the output $y[n]$ and express it in the standard form, i.e.,

$$y[n] = \sum_{k} \beta_k x[n-k] + \sum_{j} \alpha_j y[n-j]$$

(b) (10 points) Find the unit-pulse response of the system $h[n]$ using the z-transform method.

(c) (10 points) Find the zero-state response of the system when $x[n] = \left(\frac{1}{4}\right)^{n} u[n-3]$.

(d) (5 points) Find the zero-state response of the system when $x[n] = \left(\frac{1}{2}\right)^{n} u[n-5]$.

(a) \[ y[n] = \frac{1}{18} y[n-1] + 7 x[n-1] + 5 x[n] \]

(b) \[ H(z) = \frac{5 + 7z^{-1}}{1 - \frac{1}{18}z^{-1}} = \frac{z + 5z^{-1}}{z^{-1/18}} \]

\[ h[n] = 7 \left(\frac{1}{18}\right)^n u[n-1] + 5 \left(\frac{1}{18}\right)^n u[n] \]

(c) \[ Y(z) = \frac{7 + 5z}{z^{-1/18}} \cdot \frac{1}{9^3} \frac{z^{-3}}{z^{-1/18}} \]

\[ Y(z) = \frac{1}{9^3} \frac{z^{7} + 5z}{(z^{-1/18})(z^{-1/9})} z^{2} \]
\[ 9^3 z^2 y(z) = \frac{z + 5z}{(z - 118)(z - 11)} \]

\[ \frac{7 + 5z}{(z - 118)(z - 11)} = \frac{A}{z - 118} + \frac{B}{z - 11} \]

\[ A = \frac{7 + 5/18}{-1/18} = -18 \times 7 + 5 = -121 \]

\[ B = \frac{7 + 5/11}{1/11} = 18 \times 7 + 10 = 136 \]

\[ \Rightarrow 9^3 z^2 y(z) = \frac{-121 z}{z - 118} + \frac{136 z}{z - 11} \]

\[ \Rightarrow y(z) = \frac{1}{9^3} \left[ -121 \left( \frac{1}{18} \right)^{n-3} u[n-3] + 136 \left( \frac{1}{9} \right)^{n-3} u[n-3] \right] \]
Problem 2 (25 points)

For the following z-transforms:

(i) Sketch the zero-pole plot.

(ii) Assume that the sequences are right-sided (i.e., causal) and sketch the region of convergence (ROC).

(iii) Determine whether the DTFT of the sequences exists.

(iv) Compute the inverse z-transform corresponding to the determined ROC.

(a) (10 points) \( X_1(z) = \frac{z^{-2}}{z + \sqrt{3}e^{j\pi/3}} \)

(b) (15 points) \( X_2(z) = \frac{1 + \frac{1}{2}z^{-1}}{1 - \frac{\sqrt{3}}{2}z^{-1} + \frac{1}{4}z^{-2}} \)

(a) \( X_1(z) = \frac{1}{z^2(1 + \sqrt{3}e^{j\pi/3})} \)

\[ \text{double pole} : \quad z = 0 \]
\[ \text{single pole} : \quad z = -\sqrt{3}e^{j\pi/3} = \sqrt{3}e^{j2\pi/3} \]

ROC: \( |z| > \sqrt{3} \)

DTFT does not exist.

\[ z \rightarrow X_1(z) = \frac{z}{z + \sqrt{3}e^{j\pi/3}} \]

\[ X_1[n] = (\sqrt{3}e^{j\pi/3})^{-3} u[n-3] \]
(b) \[ X_2(z) = \frac{1 - z^{-1}}{4 - 3z^{-1} - z^{-2}} \]

\[ X_2(z) = \frac{z(z-1)}{4z^2 - 3z - 1} \]

\[ = \frac{z(z-1)}{(4z+1)(z-1)} \]

\[ = \frac{z}{4z+1} \]

Poles at, \( z = 1, -\frac{1}{4} \)
Zeros at, \( z = 0, 1 \)

ROC : \( |z| > \frac{1}{4} \)
DTFT exists

\[ x_2(z) = x_2[n] = \frac{1}{4} \left( -\frac{1}{4} \right)^n u[n] \]
Problem 3 (25 points)

For each of the following sequences:

(i) Determine the $z$-transform, $X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$, if it exists.

(ii) Include with your answer the region of convergence of the $z$-transform in the $z$-plane.

(iii) Specify whether or not the DTFT of the sequence, $X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega}$, exists.

(a) (10 points) $x[n] = (3)^n u[n-4] + \left(\frac{1}{3}\right)^{n-4} u[n-2]$

(b) (10 points) $x[n] = (0.2)^{n-1} (u[n-1] - u[n-6])$

(c) (5 points) $x[n] = \sin(\pi n) - \cos(\pi n)$

(a) $x[n] = 3^n (3)^{n-4} u[n-4] + 8^2 \left(\frac{1}{3}\right)^{n-2} u[n-2]$

$$X(z) = 3^n z^{-4} \frac{z}{z-3} + 8^2 z^{-2} \frac{z}{z-1/8}$$

$\text{ROC: } |z| > 3$

$$X(z) = \frac{512 z^3 - 1536 z^2 + 64}{(z-3) z^3 (8z-1)}$$

$\text{ROC: } |z| > 3$

DTFT does not exist.
(b) \[ x[n] = (0.2)^{n-1} u[n-1] - (0.2)^{n-6} u[n-6] \]

\[ = (0.2)^{n-1} u[n-1] - (0.2)^5 (0.2)^{n-6} u[n-6] \]

\[ x(z) = z^{-1} \frac{z}{z-0.2} - (0.2)^5 z^{-6} \frac{z}{z-0.2} \]

\[ = \frac{z^5 + 0.00032}{z^5 (z-0.2)} \]

ROC: \(|z| > 0.2\)

DTFT exists

(c) \[ g[n] = [\sin(\pi n) - \cos(\pi n)] u[n] \]

\[ = -(-1)^n u[n] \]

\[ x(z) = -\frac{z}{z+1} \]

ROC: \(|z| > 1\)

DTFT does not exist
Problem 4 (20 points)

Determine whether the following systems characterized by the following relations are, with respect to the input,

(i) linear or non-linear  
(ii) causal or non-causal  
(iii) shift-invariant or shift-varying

Assume that the input is zero before $n = 0$ and that the initial conditions of the systems are all set to zero. Justify your answers with proofs or counter-examples.

(a) (5 points) $y[n] = 2y[n-4] + 3x[n] + 7x[n-1]$

(b) (5 points) $y[n+1] = n^2 x[n] - 2 \cos(\pi n) y[n]$

(c) (10 points) $y[n-1] = \sum_{k=-\infty}^{\infty} x[n-k] \left( \frac{1}{5} \right)^k u[k]$

(a) LCCDE $\Rightarrow$ LTI

It is causal

(b) $y[n+1] = n^2 x[n] - 2 (-1)^n y[n]$

$y[0] = 0, \quad y[1] = 0$

Since, $n \geq 0,$

$y[n+1] = n^2 x[n] - 2 (-1)^n y[n]$

This is linear not constant coefficient difference equation (LCCDE)

$\Rightarrow \quad$ Not Linear, Not Shift Invariant

Causal!
\( y[n-1] = \sum_{k=-\infty}^{\infty} x[n-k] \frac{1}{2^k} u[k] \)

Let \( z[n] = y[n-1] \)

\[ z[n] = \sum_{k=-\infty}^{\infty} x[n-k] \left( \frac{1}{2} \right)^k u[k] \]

\[ z[n] = x[n] * h[n] \]

where \( h[n] = \left( \frac{1}{2} \right)^n u[n] \)

Since, this system follows convolution

\[ LTI \]

\[ \underline{Not\ causal!} \quad \text{(As } y[n-1] \text{ is dependent on } x[n] \text{)} \]