UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 410 Digital Signal Processing

Quiz 3

Professors Bresler, Hasegawa-Johnson — Wednesday October 6th, 2010

Solutions — **Section: 1pm (E) or 3pm(G)**

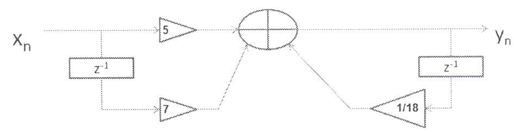
Instructions

- You may not use any calculators, cell phones, earphones, or any other forms of electronics on this quiz.
- Show all your work to receive full credit for your answers.
- When you are asked to "calculate", "determine", or "find", this means providing closed-form expressions (i.e., without summation or integration signs).
- Neatness counts. If we are unable to read your work, we cannot grade it.
- Turn in your entire booklet once you are finished. No extra booklet or papers will be considered.
- The asterisk mark ($*$) denotes convolution.

Problem	Points
1	
2	
3	
4	
Total	

Problem 1 (30 points)

Consider the causal, linear shift-invariant (LSI) system (with zero initial conditions) described in the following diagram:



(a) (5 points) Determine the linear constant coefficient difference equation relating the input $x[n]$ to the output $y[n]$ and express it in the standard form, i.e.,

$$y[n] = \sum_k \beta_k x[n-k] + \sum_j \alpha_j y[n-j]$$

(b) (10 points) Find the unit-pulse response of the system $h[n]$ using the z-transform method.

(c) (10 points) Find the zero-state response of the system when $x[n] = \left(\frac{1}{9}\right)^n u[n-3]$

(d) (5 points) Find the zero-state response of the system when $x[n] = \left(\frac{1}{9}\right)^n u[n-5]$

(a) $y[n] = \frac{1}{18} y[n-1] + 7 x[n-1] + 5 x[n]$

(b) $H(z) = \dfrac{5 + 7z^{-1}}{1 - \frac{1}{18} z^{-1}} = \dfrac{7 + 5z}{z - 1/18}$

$= z^{-1} \dfrac{7z}{z - 1/18} + \dfrac{5z}{z - 1/18}$

$h[n] = 7\left(\frac{1}{18}\right)^{n-1} u[n-1] + 5\left(\frac{1}{18}\right)^n u[n]$

(c) $Y(z) = \dfrac{7+5z}{z - 1/18} \cdot \dfrac{1}{9^3} z^{-3} \dfrac{z}{z - 1/9}$

$= \dfrac{1}{9^3} \dfrac{7+5z}{(z - 1/18)(z - 1/9) z^2}$

$$9^3 z^2 Y(z) = \frac{7+5z}{(z-1/18)(z-1/9)}$$

$$\frac{7+5z}{(z-1/18)(z-1/9)} = \frac{A}{z-1/18} + \frac{B}{z-1/9}$$

$$A = \frac{7+5/18}{-1/18} = -18\times 7+5 = -121$$

$$B = \frac{7+5/9}{1/18} = 18\times 7 + 10 = 136$$

$$\Rightarrow \quad 9^3 z^3 Y(z) = \frac{-121 z}{z-1/18} + \frac{136 z}{z-1/9}$$

$$\Rightarrow \quad Y(z) = \frac{1}{9^3}\left[-121\left(\frac{1}{18}\right)^{n-3} u[n-3] + 136\left(\frac{1}{9}\right)^{n-3} u[n-3]\right]$$

$\frac{\pi}{3} - \pi = -\frac{2\pi}{3}$

$\frac{\pi}{3} + 2\pi = \frac{4\pi}{3} - 2\pi \quad -\frac{2\pi}{3} + 2\pi$

Problem 2 (25 points)

For the following z-transforms:

(i) Sketch the zero-pole plot.

(ii) Assume that the sequences are right-sided (i.e., causal) and sketch the region of convergence (ROC)

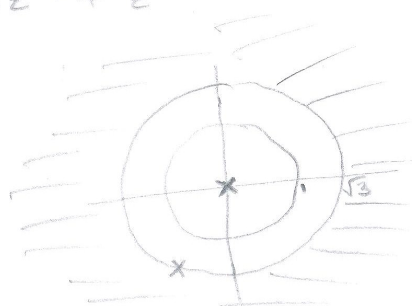
(iii) Determine whether the DTFT of the sequences exists.

(iv) Compute the inverse z-transform corresponding to the determined ROC

(a) (10 points) $X_1(z) = \dfrac{z^{-2}}{z + \sqrt{3}e^{j\frac{\pi}{3}}}$

(b) (15 points) $X_2(z) = \dfrac{1 + \frac{1}{3}z^{-1}}{\frac{1}{15} - \frac{8}{15}z^{-1} + \frac{1}{15}z^{-2}}$ $\qquad \dfrac{-1 + z^{-1}}{-4 + 3z^{-1} + z^{-2}}$

(a) $X_1(z) = \dfrac{1}{z^2 (z + \sqrt{3} e^{j\pi/3})}$



double pole : $z = 0$

Single pole : $z = -\sqrt{3} e^{j\pi/3}$

$= \sqrt{3} e^{-j2\pi/3}$

ROC : $|z| > \sqrt{3}$,

DTFT does not exist.

$z \quad z^2 X_1(z) = \dfrac{z}{z + \sqrt{3} e^{j\pi/3}}$

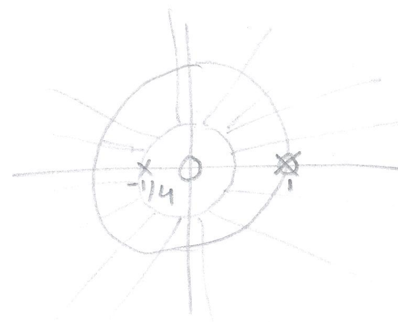
$x_1[n] = (-\sqrt{3} e^{j\pi/3})^{n-3} u[n-3]$

(b) $$X_2(z) = \frac{1 - z^{-1}}{4 - 3z^{-1} - z^{-2}}$$

$$X_2(z) = \frac{z\,(z-1)}{4z^2 - 3z - 1}$$

$$= \frac{z(z-1)}{(4z+1)(z-1)}$$

$$= \frac{z}{4z+1}$$



Poles at, $z = 1, -1/4$

Zeros at, $z = 0, 1$

ROC : $|z| > 1/4$

DTFT exists

$$\cancel{\frac{X_2(z)}{z}} = \quad x_2[n] = \frac{1}{4}\left(-\frac{1}{4}\right)^n u[n]$$

Problem 3 (25 points)

For each of the following sequences:

(i) Determine the z-transform, $X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$, if it exists.

(ii) Include with your answer the region of convergence of the z-transform in the z-plane.

(iii) Specify whether or not the DTFT of the sequence, $X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$, exists.

(a) (10 points) $x[n] = (3)^n u[n-4] + (\frac{1}{8})^{n-4} u[n-2]$

(b) (10 points) $x[n] = (0.2)^{n-1}(u[n-1] - u[n-6])$

(c) (5 points) $x[n] = \sin(\pi n) - \cos(\pi n)$

~~(a) X(z) =~~

(a) $x[n] = 3^4 (3)^{n-4} u[n-4] + 8^2 \left(\frac{1}{8}\right)^{n-2} u[n-2]$

$$X(z) = 3^4 z^{-4} \frac{z}{z-3} + 8^2 z^{-2} \frac{z}{z - 1/8}$$

~~ROC: $|z|>3$~~

$$X(z) = \frac{512z^3 - 1536z^2 + 648z - 81}{(z-3)\, z^3 (8z-1)}$$

ROC : $|z| > 3$

DTFT das not exist.

(b) $x[n] = (0.2)^{n-1} u[n-1] - (0.2)^{n-1} u[n-6]$

$$= (0.2)^{n-1} u[n-1] - (0.2)^5 (0.2)^{n-6} u[n-6]$$

$$X(z) = z^{-1} \frac{z}{z-0.2} - (0.2)^5 z^{-6} \frac{z}{z-0.2}$$

$$= \frac{z^5 + 0.00032}{z^5 (z-0.2)}$$

ROC $|z| > 0.2$

DTFT exists

(c) $x[n] = [\sin(\pi n) - \cos(\pi n)] u[n]$

$$= -(-1)^n u[n]$$

$$X(z) = -\frac{z}{z+1}$$

ROC: $|z| > 1$

DTFT does not exist

Problem 4 (20 points)

Determine whether the following systems characterized by the following relations are, with respect to the input,

(i) linear or non-linear (ii) causal or non-causal (iii) shift-invariant or shift-varying

Assume that the input is zero before $n = 0$ and that the initial conditions of the systems are all set to zero. **Justify** your answers with proofs or counter-examples.

(a) (5 points) $y[n] = 2y[n-4] + 3x[n] + 7x[n-1]$

(b) (5 points) $y[n+1] = n^2 x[|n|] - 2\cos(\pi n)\, y[n]$

(c) (10 points) $y[n-1] = \sum_{k=-\infty}^{\infty} x[n-k] \left(\frac{1}{5}\right)^k u[k]$

(a) LCCDE $\Rightarrow$ LTI

It is causal

(b) $y[n+1] = n^2 x[|n|] - 2(-1)^n y[n]$

$y[0] = 0$, $y[1] = 0$

Since, $n \geq 0$,

$y[n+1] = n^2 x[n] - 2(-1)^n y[n]$

This is linear not constant coefficient difference equation (LNCCDE)

$\Rightarrow$ ~~Not linear~~ Linear, Not Shift Invariant

Causal !

(c) $y[n-1] = \sum_{k=-\infty}^{\infty} x[n-k] \left(\frac{1}{5}\right)^k u[k]$

Let $z[n] = y[n-1]$

$\Rightarrow z[n] = \sum_{k=-\infty}^{\infty} x[n-k] \left(\frac{1}{5}\right)^k u[k]$

$\Rightarrow z[n] = x[n] * h[n]$

where $h[n] = \left(\frac{1}{5}\right)^n u[n]$

Since, this system follows convolution

$\Rightarrow$ LTI

Not causal ! (As $y[n-1]$ ~~can~~ is dependent on $x[n]$)