PREFACE

There is now considerable literature on the subject of the operational analysis of circuits, which has developed and extended the methods introduced by Oliver Heaviside in his classic Electromagnetic Theory and Electrical Papers. Yet there is no text which gathers together the various parts of the subject in a form available to engineers and physicists. This the present treatment aims to accomplish, without undue emphasis on any one aspect or point of approach.

The operational method is very powerful, and it can be just as rigorous as the classic analysis on which it is based. It should be, and undoubtedly will be, increasingly used by all who deal with circuit transients. This contemplates not only the circuits of electricity, but also those of acoustics, mechanics, thermics, hydraulics and so on. The object of this text is an exposition and substantiation of operational methods in a form applicable to all sorts of circuit problems. Illustrations are given freely and these are often electrical; but this is not a collection of interesting electrical problems, nor a treatment of transmission lines and cables.

It is entirely possible for a computer to perform the algebraic work necessary for the symbolic solution of alternating-current networks in the steady state without any grasp of the philosophy of the symbolic treatment or of the mathematics of differential equations on which it is based. The operational method bears much the same relation to transient problems that the symbolic method bears to steady-state problems. Both are shorthand processes based on classic circuit analysis. It is entirely possible to utilize the operational method on specific problems without the slightest idea of why and when it does or does not work. This, however, is computation and not analysis. In order to extend the operational method, to apply it freely and safely, to save
time by the employment of its directness and simplicity on new problems, it is necessary to have at least some grasp of the classic mathematics for which it is the working tool. This means primarily a grasp of Fourier analysis and of its treatment in terms of the complex variable. Unfortunately this branch of mathematics is often absent from the mathematical equipment of engineers. Perhaps as good a way to approach the study of Fourier methods as any is through the operational scheme. I have therefore included a brief survey of certain essential parts of the classic treatment, although of course without any pretense to completeness in this regard. On the other hand I have attempted to include all the essential features of the Heaviside type of operational analysis, and to show their dependence upon the classic processes which constitute their background. This may make the reading difficult for those students who meet certain branches of mathematics for the first time; but it will serve a useful purpose if it demonstrates the usefulness of extending their mathematical equipment along the lines indicated. After all a little mathematical knowledge can be a dangerous thing, and the use of operational methods except for pure computation should be accompanied by an appreciation of the logic of more than simple algebra.

Finally it should be emphasized that I write as an engineer, and that I do not pretend to be a mathematician. I lean for support, and expect always so to lean, upon the mathematician, just as I must lean upon the chemist, the physician, or the lawyer. It is certainly incumbent upon me, however, to learn the language and as much as I can of the processes of thought of the mathematician, to do my part to bridge the chasm that so often separates the engineer and the pure scientist. I write in the hope that this will be the attitude also of those of my readers to whom this text will be useful. Perhaps it will not be uninteresting to the mathematician to note an engineer's approach to a problem with this situation in mind.

This work has resulted from a study made during several years of presenting the subject to graduate students (principally in electrical engineering) at the Massachusetts Institute of Technology. I hope that operational analysis will gradually become part of undergraduate study as well; that this text will help this to occur; and that it will be found useful in graduate courses in other fields than electrical engineering. There are many practising engineers who may find that it is suggestive or even of direct utility. With this in mind, I have included some problems that are rather more than simple exercises.

I have made many references in footnotes to publications dealing directly with operational methods or with allied matters. This is done not only to provide a guide to the literature of the subject, but also to acknowledge my indebtedness to the large number of workers in this field.

I am much indebted to many of my students and colleagues who have labored with me as this subject has so rapidly developed in the past decade. Professor M. S. Vallarta was perhaps the first to give freely of his inspiration and aid. He in fact wrote the first set of class notes which I used. I am much indebted also to Mr. M. F. Gardner, who has recently assisted me in the teaching of the subject, and who has read the manuscript and made suggestions, many of which I fear were sadly needed. Mr. Parry H. Moon and Mr. Nathan Howitt have also helped me enormously in corrections and with the proof, problems, and figures. I have made other acknowledgments in the text. Still more, however, I am indebted to Professor Norbert Wiener, of the Department of Mathematics of the Massachusetts Institute of Technology. He has patiently guided me around many a mathematical pitfall, and if I have still erred and fallen into some such, I wish to definitely go on record that it is not his fault. Moreover he has written an appendix to this text on certain mathematical points. I did not know an engineer and a mathematician could have such good times
together. I only wish that I could get the real vital grasp of the basic logic of mathematics that he has of the basic principles of physics. Finally I wish to express to Professor D. C. Jackson, the head of my department, and to President S. W. Stratton, of the Massachusetts Institute of Technology, my appreciation of the encouragement on their part which makes this sort of endeavor possible and pleasant.

V. Bush

CAMBRIDGE, MASS.
Jan., 1929.

NOTE TO FIRST EDITION, SECOND PRINTING, CORRECTED

During several years of class use of this text a number of minor errors have appeared which Professor M. F. Gardner has kindly rectified in preparing this edition for reprinting.

V. Bush.

December 10, 1936.

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OPERATIONAL CIRCUIT ANALYSIS

CHAPTER I

THE CIRCUITS OF ENGINEERING AND PHYSICS

A surprising amount of engineering analysis is concerned with circuits of various sorts. Any one who can analyze circuits with facility under all sorts of conditions has mastered a very important part of engineering technique. This applies not alone to electrical engineering, where the universal use of circuits is strikingly apparent, but also to other branches; for there are circuits of hydraulics, mechanics, thermics, acoustics, and even chemistry. We evidently first need to define what is meant by a circuit, and to specify the properties of those which we shall here treat operationally.

A circuit may be defined as a physical entity in which varying magnitudes can be sufficiently specified in terms of time and a single dimension. It thus involves movement or variation along a path. This is contrasted with the field problem in which there is variation with two or three dimensions. Thus the analysis of the behavior of a speaking tube is a circuit problem, while a study of the production of sound in a room by a loud speaker is a field problem. In the speaking tube the air pressure and velocity may be assumed with sufficient engineering accuracy to be the same at a given instant for all points on a cross-section of the tube. It is hence possible to specify what is happening in the tube by specifying either the pressure or the velocity as a function of the time and a single dimension — the distance along the tube. In the loud-speaker problem, however, a description is necessary which involves every point of the room. This requires three dimensions, and a field problem is encountered.
Circuits may be either closed or open. With a closed circuit we have a cyclic space arrangement such that the path is continuous, and by proceeding along the circuit we can return to the starting point. A magnetic circuit is always closed for magnetic flux lines are always continuous. The path may vary in cross-section or in nature of material. Thus in the magnetic circuit of a dynamo one part may be the iron of the armature, another the air gap, and others the pole pieces and yoke; and the magnetic flux passes in a closed path through all these parts in series. As an example of an open circuit, consider a long copper rod one end of which is thrust into the fire. This is a thermal circuit, and the temperature and rate of heat flow are substantially functions of the single dimension of distance along the rod. Yet there is evidently no closed path and we have an open circuit.

A network is a combination of similar interrelated simple circuits. It is usual to speak of networks only when the interrelated circuits are of the same kind. Thus an electric bell connected to a dry battery gives an electric circuit, and two bells connected to the same battery give a network. But in the electric bell is a magnetic circuit necessary to its operation. The electric and magnetic circuits of the bell are interrelated, but the combination is not called a network. The number of degrees of freedom of a network is equal to the number of simple circuits composing it, or the minimum number of separate values of a variable necessary to specify its performance. The separate circuits of a network are often called branches or meshes. Usage is not fixed, and circuit is often used to mean a network with several branches.

2. Circuit Problems and Field Problems. Circuit analysis is perhaps the most important sort of technical analysis used by engineers, and especially by electrical engineers. Engineers have plenty of field problems, and would much like to be able to analyze them readily and completely. Take, for example, the problem in aeronautics of the flow of air past airfoils, or the problems of ship resistance and wave formation, or the flow in steam or hydraulic turbines. Unfortunately it is usually possible to proceed only a short distance by formal analysis in problems such as these. Wherever the study can be reduced to involve one dimension only, it is much more amenable to analysis. Thus the hydraulic problem of the flow of water in a penstock and nozzle becomes manageable as a circuit problem, even for the transient conditions of flow involved in some water-power problems.

The evolution of electrical engineering has proceeded at such an extraordinary rate primarily because its technical problems have been such as to yield to analysis. This has made possible the exact design of apparatus and the accurate prediction of its performance. Fundamentally this has been true because most electrical engineering technical problems involve circuits, because these circuits can be accurately represented as such, and because the results of electric circuit analysis are readily checked by measurement. Electric machinery and other paraphernalia are essentially assemblages of interrelated circuits. The relationships involved electrically are very accurate, and it is much easier to insert an ammeter into an electric circuit than, for example, to put a flow meter into a pipe line.

For this reason circuit analysis has been developed primarily at the hands of electrical engineers, and the terminology of circuits is usually their terminology. This procedure will be continued in this text, but we shall keep in mind that other types of circuits are also important.

As electrical engineering develops, it becomes necessary to analyze new types of circuits. The circuit problems become more complex and inherently more difficult. Thus new methods of attack are necessary and are being developed, and relatively old methods are being polished into form for rapid use. The greatest tool for this purpose that is now being developed is the Heaviside Operational Calculus.
The present text is primarily devoted to the presentation of this tool and allied matters in form for engineering use. Operational methods have so far been applied only to circuit problems. There is no reason why they should not be very useful also in connection with field problems, but so far they have not been developed for this purpose.

Unless the power of engineering analysis keeps pace with the growing difficulty of its technical problems, rapid engineering progress will stop.

3. Lumped and Distributed Parameters. The characteristics of a circuit are specified in terms of circuit parameters or constants. The former is the preferable term, for sometimes the parameters are not fixed in value and it is confusing to call them constants. The inductance of an electric circuit, the weight of water in a unit length of pipe, the heat-storage capacity of a unit block of metal through which heat is flowing may all be circuit parameters.

Parameters may be either lumped or distributed. With a simple circuit containing only lumped parameters, one variable at least is a function of time only; that is, it has the same value at a given instant at all parts of a circuit. This is true for the flow of an incompressible fluid in an unyielding pipe. The parameters of circuits are never strictly lumped, but often it is sufficiently accurate to consider them so. Thus an inductance coil cannot be built without distributed capacitance, nor a condenser without a small amount of inductance in its leads.

When the parameters are distributed, there may be a space variation as well as a time variation of the variables in the circuit. Traveling waves can occur in the system. An electric transmission line, in which each element of the line may be assigned values of inductance and capacitance, may have at a certain instant many amperes at one point in the wire, and at the same instant no current at all in the same wire and only a few feet away. Suppose we pump water with a single-acting pump through a long length of very flexible rubber tubing. The water will proceed in a series of pulses or waves, and at a given instant the flow of water at certain points may actually be backwards toward the pump.

The methods of analysis we shall consider may be applied both to circuits with lumped and with distributed parameters.

4. Fixed and Variable Parameters. The parameters of a circuit may be either fixed or variable. In case all the parameters of a circuit are fixed, the circuit is said to be linear; and, as we shall see, the principle of superposition may then be used. A fixed parameter is of course one which has the same value at all times, and irrespective of what is being done to the circuit at the moment. Circuit parameters are never absolutely fixed in any practical problem, but it is often very accurate to assume them so. It is usually quite justifiable to assume the resistance of a copper wire carrying a moderate current to be constant over a brief period of time. The same assumption in regard to the filament of an incandescent lamp may lead to gross errors.

Many parameters are substantially constant within certain limits. The leakage of an aerial transmission line is nearly zero until corona appears. When a steel rod is tapped on the end, the traveling sound waves in the metal may be analyzed by considering a circuit with fixed parameters, unless the elastic limit of the material is reached at some point.

When parameters vary, they may be functions of various things. The mutual inductance of a stationary and a rotating circuit in an alternator is a function of the time, and often closely a sinusoidal function of the time. The internal resistance of a rectifier is a function of the direction and amount of current through it. The resistance of a thermionic tube is ordinarily a function of both the grid and plate voltages, although in the shielded-grid tube it may be a function of the grid voltage only. With an iron-cored coil the inductance is strictly a function of the entire past history of the current, although often closely enough repre-
sentable as a function of the current only, or even as a con-
stant.

The analysis of circuits with fixed parameters is much
simpler than that of circuits with variable parameters. This
text will not attempt any extended operational treat-
ment of the latter, as operational methods have not as yet
been successfully applied in practice to circuits with vari-
able parameters. In Chapter XV, however, we shall
outline the approach to the problem.

5. Idealized Circuits. All practical circuits must be
idealized before they can be analyzed. Various approxi-
mations are introduced for simplification. The extent to
which it is legitimate to proceed in this manner is dictated
only by experience. Strictly, the accuracy of the mathe-
matical solution of a physical problem can be determined
only by experiment. Yet experience with similar problems
will often tell us a priori about what approximations may be
safely introduced.

Often a circuit problem is analyzed when actually a field
problem exists. All electrical circuit problems are of this
nature, although the approximation may be very close in
nearly. In analyzing power circuits at commercial fre-
dencies we do not hesitate to neglect the fact that every
alternating-current circuit radiates energy, for a simple
computation will show that the amount of energy thus radi-
ated is negligible. To do the same with the circuit of a
short-wave radio receiver is to be wide of the mark. In
fact, when the frequency is so high that the corresponding
wave length is small compared to the physical dimensions of
the apparatus, any treatment by means of circuits is utterly
inaccurate. It is often customary, and in fact necessary if
analysis is to proceed, to assume parameters fixed which
are really variable, or lumped which are really distributed.
Experience must be our guide in this matter. This empha-
sizes the fact that it is imperative for the engineer to see
clearly the physical aspects of his analytical work at all
times. Every formula, every step should have for him a real
and vital meaning in terms of copper and iron, the flow of
water, or whatever he may treat.

Oftentimes there is a temptation for the engineer, dealing
thus with approximations and guiding his every step with
experiment, to be unduly careless of mathematical rigor.
There is no excuse for carelessness in this respect any more
than it is tolerable to use careless computations on important
engineering works. Still it is not well to pay so much atten-
tion to questions of rigor that there is neither time nor effort
left for anything else. Oliver Heaviside expressed himself
so forcibly on this matter that further comment is hardly
needed here, especially as it is improbable that any one will
use this text who does not frequently turn for inspiration
and background to Heaviside's own works, of which this is
in some sense an interpretation.

PROBLEM ON CHAPTER I

Prob. 1-1. In the following cases note whether the parameters are
actually fixed or variable, lumped or distributed. Where it would be
reasonably possible, to make assumptions, note what they are. Note whether
the circuit problem or a field problem actually exists.

a. A switch is turned, lighting the headlights of an automobile. Find the current variation.

b. A steel ball is being heat treated, and is heated uniformly to a
moderate temperature and then dropped into oil. Find the temperature
distribution and variation.

c. The gates of a lock are suddenly opened when there is a difference
of water level in the lock and the canal. Find the flow.

d. The suspension cable of a bridge is hung in place during erection
with a large sag, and then slowly hauled in at one end. Find the stresses.

e. During this process there is an accident, the cable slips back a few
feet, and then is suddenly arrested. Find the stresses.

f. A large dish of water is evaporating in still air. Find the distribu-
tion of density of water vapor near the surface.
CHAPTER II
THE FUNDAMENTAL EQUATIONS AND CIRCUIT ANALOGIES

In analyzing circuits we seek to find the effect due to some known cause, or set of causes. In an electrical circuit the nature of the applied electromotive forces may be known and the resulting currents may be desired. In a hydraulic problem the pressures impressed on the system may be known, and an attempt made to find the flow of fluid. In a heat problem we are often acquainted with a given set of temperatures and search for the rates of heat transfer. For the present, we shall assume in each case that we know exactly how the cause varies with the time. Cases where the intensity of the cause varies with the amount of effect produced are difficult, but can often be treated by methods applicable to circuits with variable parameters.¹

Sometimes the roles played by the variables representing the usual cause and effect are interchanged, but the methods of analysis need be but little altered on that account. Thus, while it is usual to impress an electromotive force on an electrical circuit, it is sometimes desirable to impress a charge or a current. We are familiar with electric sources of substantially constant voltage, such as power mains and batteries, so that the use of these to cause current gives the usual case. The Thury system, however, is an illustration of a constant-current power source. With such a system, a motor is taken out of service by short-circuiting it and put into circuit by opening a switch across it. Thus a substantially constant current is suddenly impressed on the motor. This becomes the cause, and the resulting voltages are the effect.

¹ Chapter XV.

In heat-flow problems we often meet the case of a known impressed supply of heat producing temperatures for which we must analyze. Thus, when an electric flatiron is connected in circuit, the heat flow in calories per second is fairly constant and accurately known, and analysis will show the consequent distribution of temperatures in the metal.¹

It makes little difference in method what type of circuit is analyzed or which variable is the cause. For simplicity of notation and language we shall usually treat the electric circuit, and consider as the cause a known set of applied electromotive forces which we can write as functions of the time. The method can then be carried over to other problems by analogy.

We shall thus write e for the applied electromotive force. This is a known function of the time:

\[ e = f(t). \]

The same symbol may stand for an applied temperature or pressure difference. Similarly, i represents the current, but may also mean rate of heat flow, velocity of a fluid, etc. When usual cause and effect are reversed,

\[ i = f(t) \]

is the known functional relation.

In a magnetic circuit, an applied magnetomotive force and a consequent magnetic flux must be considered. Since there is only a single parameter, reluctance, in a magnetic circuit, the entire performance is given by a law of the form of Ohm’s law except where iron is present and the reluctance is variable. Special methods are then used, which need not be entered into here.

2. Resistance. The simplest circuit parameter is that which causes dissipation and which is called the resistance. It directly opposes the current in an electric circuit, con-

¹ This is discussed in greater detail in Chapter XIV.
verted electric energy into heat. It is defined by Ohm’s law,
\[ e = Ri \]  \hspace{1cm} (1)

which states that for a simple resistance the ratio at any instant of voltage across it to current through it is given by the resistance. If the voltage is in volts and the current is in amperes, the resistance will be given in ohms.

In heat flow appears the exactly analogous thermal resistance. Its reciprocal is thermal conductance; and the thermal conductance of a unit cube is the thermal conductivity of the material. If in the above equation \( i \) represents the heat flow in calories per second and \( e \) is the temperature difference in degrees centigrade, \( R \) will be the thermal resistance as obtained from the thermal conductivity and the dimensions of the block of material, and will be measured in degrees per calorie per second.

In hydraulic systems, the parameter becomes fluid resistance. For flow through a pipe, \( R \) represents the pressure drop (in whatever units are being used) when the fluid is flowing steadily through the pipe at unit velocity.

In mechanical systems the analogous quantity is friction, and \( R \) will here represent the mechanical force necessary to maintain steady unit velocity.

When we write \( R \) we may mean any one of these parameters, but it will be simpler to confine ourselves to the single letter and to call it always the resistance. The same will be done with other parameters.

When the parameters of a circuit are lumped, \( R \) will represent the total resistance of a unit or element. With distributed parameters, we shall use the resistance of a unit length of the circuit and call this \( r \). The same procedure will apply to the other parameters as well.\(^1\)

3. Inductance. The inertia parameter of an electric circuit is its inductance. Just as resistance opposes a current, so inductance opposes a change of current. In other words, a large inductance requires a relatively large voltage across it in order to produce a given rate of change of current through it. Inductance is defined by the relation:
\[ e = L \frac{di}{dt} \]  \hspace{1cm} (2)

which states that the voltage across a pure inductance is proportional to the rate of change of current through it, the proportionality factor being the value of the inductance. Inductance will be in henries when current is in amperes and potential difference is in volts. Of course either system of c. e. s. units may be used instead of the practical system.

In mechanical problems the inertia parameter is the mass. It is equal to the force required to produce unit rate of change of velocity — that is, unit acceleration. \( L \) may then stand for the mass in pounds or grams. As before, a small letter \( l \) will stand for inductance per unit length. In a fluid circuit, hydraulic or acoustic, \( l \) will be the mass of material in unit length of the circuit or pipe, and thus will be equal to the pressure difference per unit length due to accelerating the fluid at unit rate.

There is no detectable inertia parameter in heat flow. In the slow flow of a viscous fluid it may often be neglected. In electric cables, for slow rates of current change, it can also be left out of consideration in first approximations. These circuits are closely analogous.

4. Capacitance. The spring parameter of an electric circuit is its capacitance. Just as inductance opposes a change of current, so capacitance opposes a change of voltage. In other words, a large capacitance requires a relatively large current through it in order to produce a given rate of change of voltage across it. Capacitance is then defined by the equation:
\[ i = C \frac{de}{dt} \]  \hspace{1cm} (3)

\(^1\) See Appendix A for a list of symbols used throughout this text.
An element has unit capacitance when unit rate of change of voltage across it corresponds to unit current through it. Since cause and effect are here reversed from the familiar order, we usually write

\[ e = \frac{1}{C} \int i \, dt \quad (4) \]

or

\[ e = \frac{q}{C} \quad (5) \]

and state that the capacitance of an element is given by the charge required to produce unit change of voltage across it. If the current is in amperes, the potential difference in volts, and the charge in coulombs, the capacitance will be in farads.

By prefixing an "ab" to each of these units, we have the statement for the c.g.s. electromagnetic system; while by using a "stat," we get it for the c.g.s. electrostatic system of units.

The thermal capacitance is the charge in calories, or other heat unit, that must be communicated to the element to produce unit temperature change.

In fluid flow, the capacitance of an element is given by the quantity of fluid to produce unit pressure change. When the fluid is very compressible, as is air, the meaning is apparent; for it is easy to visualize the amount of air that must be pumped into an element such as a chamber in a pipe line to raise its pressure one unit, a pound per square inch, for example. The same holds for liquids also, but here the compressibility is so small that yielding of the walls of the containing vessel contributes largely to the total capacitance. The capacitance \( c \) per unit length of a hydraulic penstock may be readily computed from the known compression constants of water and the elastic properties of the pipe.

5. Parameters of Mechanical Vibrating Systems. The three parameters, \( R \), \( L \), and \( C \) are not the only parameters which may be set up to describe the properties of circuits.

However, we shall deal with circuits only when they can be specified completely in terms of these three. When the parameters are constant, there is no doubt of the convenience of this procedure.

Our definition of a circuit is rather broad and includes mechanical systems not usually regarded as circuits; but, on account of closeness of analogy, it is convenient to so treat them. Thus, a rigid pendulum is a system, the properties of which are readily described by these three parameters; and its motion is describable as a function of time and a single dimension. For a pendulum, \( e \) may represent an impressed torque, say in dyne-cm.; \( i \), an angular velocity in radians per second; \( R \), a damping torque in dyne-cm. per radian per second; \( L \), a moment of inertia in gram-cm.\(^2\); and \( C \), a spring or gravitational factor in radians displacement per dyne-cm. restoring torque. All simple rigid mechanical oscillating systems of a single degree of freedom may be thus included in our treatment, together with coupled combinations of such systems. This includes such systems as oscillograph vibrators, galvanometers, telephone diaphragms, engine flywheels, vibrating shafts, and deflecting beams.

6. Mutual Parameters. So far we have dealt only with single circuits and the self-parameters of such circuits. In networks consisting of several circuits coupled together we have mutual parameters. Two circuits are coupled when they have an element in common, and this element may involve any one of the three parameters or any combination of them. Circuits will be denoted by Arabic numerals, as \( 1, 2, 3, \ldots \); and parameters will be numbered with corresponding subscripts, as \( R_1, L_2, \ldots \) (See Fig. 1). Mutual
parameters will have two subscripts, as \( R_{12} \), which shall be defined as the voltage produced in Circuit 1 when open by unit current in Circuit 2.\(^1\) When a circuit contains more than one parameter of the same kind, the total parameter of the circuit will be denoted by a double subscript, as \( R_{12} \) or \( R_{11} \) (\( R_{11} = R_1 + R_{12} \)).

In Fig. 1 is shown the simplest case — two electrical circuits with a conductive coupling by means of a simple resistance. The total self-resistance of Circuit 1 is equal to the voltage due to steady unit current in this circuit alone; that is, with no rate of change of current in the inductance and no charge on the condenser. This is evidently \( R_{11} = R_1 + R_{12} \). Similarly, for Circuit 2 we have \( R_{22} = R_2 + R_{12} \). The mutual resistance, \( R_{12} \), is equal to the voltage produced in 1 when steady unit current flows in 2, Circuit 1 being open. The same result is obtained for the voltage in 2 when unit steady current flows in 1. We shall call this \( R_{21} \). The relation is reciprocal, and we can express the fact thus:

\[
R_{12} = R_{21} \tag{6}
\]

The same holds true for other parameters. The coefficient of coupling is defined as the mutual resistance divided by the geometric mean of the self-resistances, or

\[
\frac{R_{12}}{\sqrt{R_{11} R_{22}}} \tag{7}
\]

When the entire resistance is common, the coefficient of coupling is unity. The same terminology holds for coupling through other parameters.

\(^1\) A uniform system of double-subscript notation will be used throughout this book, not only for the circuit parameters \( R_{ab} \), \( L_{ab} \), \( C_{ab} \), \( Z_{ab} \), but also for the minor \( M_{ab} \), for the impedance function \( Z_{ab} \) for the indicial admittance \( A_{ab} \), and for the indicial impedance, \( V_{ab} \). In all cases we shall observe the convention that the first subscript, \( a \), shall refer to the circuit in which voltages are being considered, while the second, \( b \), shall refer to the circuit in which currents are being considered. In other words, \( a \) is related to a voltage; \( b \) to a current.

With a common inductance, as in Fig. 2, we have inductive coupling. Evidently \( L_{11} = L_1 + L_{12} \), etc. as before.

Inductive coupling is possible when two coils have a common magnetic flux even when there is no actual wire common to the two circuits. (Fig. 3.) The definitions still hold.

The self-inductance of Circuit 1 is equal to the voltage resulting from unit rate of change of current in 1, with Circuit 2 open or absent, and the same for the other circuit. The mutual inductance is the voltage in 1 resulting from unit rate of change of current in 2, or vice versa. In a practical case these constants are of course determined by examination of the design or by actual test. The mutual inductance will be denoted by \( L_{12} \).

We can also have condensive coupling, as, for example, in Fig. 4. The self-capacitance of Circuit 1 is the value of charge we must pass through 1 to produce unit voltage in Circuit 1, Circuit 2 being open. This is seen to be

\[
C_{11} = C_1 + \frac{C_1 C' C''}{C_2(C' + C'') + C' C''} \quad \text{Similarly for } C_{22}. \tag{8}
\]
The mutual capacitance is the charge which must be passed through 1 to produce unit voltage in 2, circuit 2 being open, or vice versa. This is

\[ C_{12} = C_1 + C_2 + C_1C_2 \left( \frac{1}{C'} + \frac{1}{C''} \right) \]  

(9)

![Diagram of Network with Condusive Coupling](image)

**Fig. 4.** Network with Condusive Coupling.

It is also possible to have condusive coupling between circuits when they have a common electrostatic field, and without any metallic connection between them. In Fig. 5 the circles may represent either spheres or cross-sections of parallel cylindrical conductors. The self and mutual capacitances are defined exactly as before, the potentials being computed from the configuration of the electrostatic field.

![Diagram of Circuits with Common Electrostatic Field](image)

**Fig. 5.** Circuits with Common Electrostatic Field.

In the above figures we have shown in each case only two circuits with a single common parameter. There may be many individual circuits or branches in a network, and the branch between any two may contain any or all of the three types of coupling.

It is also possible to have networks of other sorts of circuits than electrical. In an alternator there are as many magnetic circuits as there are poles, and they are coupled by a common reluctance in parts of each circuit. Usually, due to symmetry, it is only necessary to compute one circuit.

If a spring is connected between two pendulums we have a condensively-coupled network. The mutual capacitance is then the angular deflection of one which will produce unit torque on the other.

Air pipes and chambers may be interconnected to form an acoustic network. Acoustic filters, which are selective to frequency, have been built in this manner. The analysis of such systems is analogous to that of electric filters.¹

Interconnected thermal circuits may also be considered.

In all these cases the analysis proceeds in exactly the same manner, the interpretation of the equations being clear if the definition on which they are set up is clear.

7. **Fundamental Circuit Laws.** Kirchhoff's Laws enable us to write the circuit equations. The first law states that the sum of all the voltages about a closed circuit is zero. This follows directly from the conservation of energy. All voltages must be included: applied electromotive forces, voltages due to the effect of self-parameters, and those induced by coupling with neighboring circuits.

The second law states that the sum of the currents flowing into a junction point is zero, and follows from the conservation of electricity. We shall consider the positive direction of current in each circuit as clockwise. In a common element the second law then enables the resulting current to be written with due regard to sign.

8. **Operational Terminology.** We shall use operational notation from the outset. Instead of

\[ e = \frac{di}{dt} \]  

(10)

let

\[ e = Li \]  

(11)


where \( p \) is simply another way of writing \( \frac{d}{dt} \). There is nothing new in this, of course.

We are quite used to operators and operands. An operator is a symbol which states that a certain operation is to be performed on the quantity to which it applies and which is called the operand. Thus, in the expression,

\[
y = \cos x
\]

the \( \cos \) is a symbol for a certain familiar operation, and \( x \) is the operand. The matter would be much simpler if our notation were consistent, but, like many similar things which just grew, it is far from it. In the expression,

\[
y = \log x
\]

the operator and operand are clear. But in the inverse expression,

\[
x = e^y
\]

it is not nearly so evident that we denote an operation on \( y \). Often a unit negative exponent means the inverse operation. Thus, in

\[
x = \cos^{-1} y
\]

the \( \cos^{-1} \) is the inverse operator to \( \cos \), meaning that when they are applied together their effects annul. Thus,

\[
x = \cos^{-1} \cos x = \cos \cos^{-1} x.
\]

But while in the expression,

\[
x^2
\]

the exponent 2 means to perform the operation of squaring,

\[
x^2
\]

denotes the reciprocal, not the inverse, which latter is written:

\[
x^3
\]

We do not get confused by these things because of long familiarity, however much we may have struggled in early days. It is unfortunate that operational notation is so complicated; but we are not responsible for it, and the best we can do is to try to be simple and consistent in any new operators we may introduce.

In setting up the differential equations of a network, we deal with the derivative operator \( p \), and various combinations of such operators which are always applied to continuous functions of time.\(^1\) \( p \) is here simply an abbreviation for \( \frac{d}{dt} \); and, when applied to a continuous function of the time, it takes the derivative or slope of that function. When the operational solution of networks is dealt with, we shall

\(^1\) A large amount of work has been done on operators and operational methods. We shall refer to much of this literature at points where it particularly applies or in order to acknowledge the source of material included in the text. At this point we note a few papers which give widely different methods of approach to the subject, to some of which we shall again refer later on.


T. J. I’A. Bromwich, Examples of Operational Methods in Mathematical Physics, Phil. Mag., 37, 1919, p. 497.


P. Lévy, Leçons d’Analyse Fonctionnelle, Gauthier, Paris, 1922.


Pleijel and Liljeblad, Operatorkalkylens Samband med den Symboliska Metoden, Teknisk Tids., 1919, p. 25.


introduce boundary conditions, and shall find our operators acting on discontinuous functions. The operator \( p \) is then no longer interpretable simply as a derivative operator. We shall, in fact, interpret it as we evaluate it; so that our procedure at first is heuristic or experimental, and definitions will follow later. These definitions and laws governing operators might be set up in various ways and still lead to the correct solutions of problems. The desideratum is of course to set up a system which is as simple and convenient as possible, and which will yield correct results. As we introduce operational processes we shall continually bear this in mind. The inverse operator will be written \( p^{-1} \). \( p^{-1} \) is then defined as that operator which annuls the effect of \( p \), so that

\[
p^{-1}pf(t) = f(t). \tag{12}
\]

The operation \( p^{-1} \) is thus an integration, since it cancels the effect of a differentiation. Just what sort of integration it is depends on circumstances, as will be shown later.

All of algebra depends upon the associative, distributive, and commutative laws, which may be grouped and called the algebraic laws. If \( a, b, \) and \( c \) are algebraic quantities, these laws state that

\[
a(bc) = (ab)c
a(b + c) = ab + ac
ab = ba. \tag{13}
\]

When quantities obey these laws, and only then, all the usual algebraic transformations may be applied.

In the theory of differential equations it is shown that the operator \( p \) obeys the laws of algebra when it is applied to functions that are continuous.

Many operators are used in mathematics which do not obey these laws, and which cannot, therefore, be combined by the usual algebraic rules. In establishing our operational processes we shall at first simply assume that the operators we utilize obey these laws, and later we shall justify this assumption.

Using the exponent notation,

\[
p^{n}f(t) = p^n f(t)
p^m p^n f(t) = p^{m+n} f(t), \text{ etc.}
\]

We shall use the functional notation, often using fairly complicated combinations of \( p \) with various constants. \( F(p) \) may mean such an expression containing \( p \). Various operators may be distinguished by subscripts. When an operator \( F(p) \) is applied to a time function \( f(t) \) it generates another time function. Such a function, \( F(p) \), is therefore called a generating function.

Note one special point: the expression,

\[
F_1(p)f_1(t) \cdot F_2(p)f_2(t)
\]

is equal to

\[
F_2(p)f_2(t) \cdot F_1(p)f_1(t)
\]

for it is a product of two time functions. The order cannot be changed in any other way without altering the meaning.

9. Differential Equations of Lumped Networks. For a single circuit with fixed lumped parameters, not coupled to other circuits, Kirchhoff's first law states that

\[
R_i i_1 + L_i p^2 i_1 + \frac{1}{C_i p} i_1 = e_1. \tag{14}
\]

The successive terms are the voltages produced by \( i_1 \) in conjunction with resistance, inductance, and capacitance; while \( e \) is the applied electromotive force which is assumed to be a known function of the time:

\[
e_1 = f_1(t).
\]

If this is abbreviated by writing

\[
R_i + L_i p + \frac{1}{C_i p} = z_1(p) \tag{15}
\]

the equation becomes

\[
z_1(p)i_1 = f_1(t).
\]
times carelessly used when parameters are not fixed, thus producing some rather startling results.

A network with fixed parameters is called a linear network, and the equations which describe it are known as linear differential equations. The theory of such equations shows why superposition is legitimate with a linear network. The reason can also be visualized physically. If a parameter always has the same value, no matter what may be occurring in the circuit at the time, then a single impressed force will arouse the same reaction no matter what other forces may be acting simultaneously. Thus the result of several forces acting simultaneously is found by adding the effects which would be produced by each acting alone.

In the differential equations of a linear network this means that there is no loss of generality if only one impressed force is considered, noting that if there are several, we shall be able to thus treat each in turn for a complete result.

Therefore,

$$\begin{bmatrix}
z_{11}(p)i_1 + z_{12}(p)i_2 + \cdots + z_{1n}(p)i_n = f_1(t) \\
z_{21}(p)i_1 + z_{22}(p)i_2 + \cdots + z_{2n}(p)i_n = f_2(t) \\
\vdots \\
z_{n1}(p)i_1 + z_{n2}(p)i_2 + \cdots + z_{nn}(p)i_n = f_n(t)
\end{bmatrix}$$

(17)

These equations, together with a description of the initial condition of the network, completely specify the network performance.

10. **Principle of Superposition.** When several electromotive forces are simultaneously applied to a network of fixed parameters, each produces its own effect independently of all the rest. The effect of each alone may be calculated independently and the values added to obtain a true result.

This procedure is continually utilized by engineers and physicists, who analyze for the effects of separate causes and superpose these effects to obtain a resultant due to the simultaneous action of the various causes. This method is valid only when all parameters, $r$, $l$, and $c$ are strictly fixed either absolutely or as functions of the time.\footnote{The use of superposition when variable parameters which are functions of the time are present will be mentioned in Chapter XV, Sec. 2.}

\[ EQUATIONS \ AND \ CIRCUIT \ ANALOGIES \]
The signs are in accordance with the convention of positive clockwise current in each circuit. They are best kept straight by noting the relative way in which the current flows through mutual elements. If $i_2$ had been taken counterclockwise, all signs would have been plus. It makes no difference in the final result which system is adopted.

When two circuits are inductively coupled by a transformer, there is an opportunity for confusion in signs. Especially is this true when three circuits are thus coupled in pairs by three transformers. It then pays to actually trace out the connection of the windings and note the direction of the voltages induced when the current in each circuit in turn has a positive rate of increase.

The above circuit equations hold exactly as written for the mechanical system of Fig. 7. The flywheels are loose on the shaft; and each is attached to a helical spring, the other end of which is fixed. A third spring connects the two flywheels. Any known torque variation may be considered applied at $e$, and this is $f(t)$. $L_1$ and $L_2$ are the moments of inertia of the two flywheels. $R_1$ and $R_2$ are the frictional torques of each at unit angular velocity. $C_1$, $C_2$, and $C_{12}$ are the spring constants of the three springs; that is, the deflection in radians to produce unit restoring torque. $i_1$ and $i_2$ are the angular velocities of the two wheels. The equations assume that all parameters are fixed, which may be seriously in error with regard to the frictional terms.

In Fig. 8 is shown a rather artificial hydraulic network which has the same equations. $L_1$ and $L_2$ are the total moving masses in the two pipes, and $R_1$ and $R_2$, the corresponding fluid friction parameters. The applied force $f(t)$ is exerted on the piston. The capacitances correspond to the spring constants of the three diaphragms. Voltages correspond to pressures, and currents to rates of fluid flow.

The only safe way to set up analogies between different sorts of circuits is to actually write the differential equations and identify the meaning of each term. This also serves to keep the matter of units clear.

1 For a treatment of a practical application of mechanical analogues, see Maxfield and Harrison, Methods of High Quality Recording, Trans. A.I.E.E., 45, 1926, p. 334; B.S.T.J., 5, 1926, p. 493.


PROBLEMS ON CHAPTER II

Prob. 1-2. Draw a network consisting of two circuits coupled together inductively, conductively, and electrostatically. Make the network as simple as is possible under these conditions.

Prob. 2-2. A crude seismograph is built by attaching an iron ball to a rod, mounting the rod horizontally so that its end pivots against the top of a wall, and then supporting the whole by a wire fastened to the rod near the ball and running at an angle of 45° to a screw eye in the wall in a vertical plane through the center of the rod. A smoked plate is arranged to detect any relative motion of the ball and the floor. An earthquake moves the building as a whole, and from the record produced it is desired to show what was the component of earth motion perpendicular to the rod. Write the differential equations. What is the analogous electric circuit problem? Identify each parameter, and note its exact definition.

Prob. 3-2. A radioactive substance A changes into a second substance B, which in turn changes into C, which is a permanent element. The rate at which A transforms is proportional to the amount of A present, and the same is true for B. Required, the laws of change. Write the differential equations. Set up an analogous problem in hydraulics. Also discuss the electric circuit analogue.

Prob. 4-2. For which of the following electric circuits will the principle of superposition hold strictly, or approximately, and within what limits?
   a. A direct-current generator supplying the filament of an incandescent lamp.
   b. A filter constructed of air-core coils and paper condensers, for use at audio frequencies.
   c. A telephone circuit consisting of microphone, battery, line, and receiver.
   d. A 220-kv power transmission line with its terminal transformers.
   e. An arc-type electric furnace.

Prob. 5-2. A telephone line with wires spaced one foot horizontally runs for a mile parallel to a power transmission line. The three wires of the power line are spaced 6 feet in an equilateral triangle, the base of which is horizontal and 30 feet high. The telephone loop is 10 feet high, and its center line is 12 feet horizontally from the center line of the power line. During trouble on the power line a large single-phase current flows in the two wires nearest the telephone line, the current in the third wire being negligible. It is required to find the effect produced by interference on the telephone line, when the latter is terminated at the ends of the mile exposure by being (a) open, (b) short circuited. What parameters are involved, and how would they be computed?

Prob. 6-2. A network is made up of three circuits, each containing all three parameters, and every circuit being inductively coupled to each of the other two. Set up the differential equations, with particular regard to signs.

Prob. 7-2. A pendulum is hanging at rest from a support which is suddenly displaced a short distance horizontally. What is the analogous problem of electric circuits?

Prob. 8-2. Two pendulum bobs are coupled by means of a spring having a stiffness factor $\frac{1}{c}$ of 200 dyne-cm/radian. The masses of the pendulums are 100 g. and 300 g., respectively, while their lengths are 100 cm. and 50 cm. A steady force of 100 dynes is suddenly applied to the 100-gram bob in such a direction as to tend to separate the two pendulums. Neglect friction and also the mass of the bob suspensions. Write the differential equations and give the analogous electric circuit.

Prob. 9-2. Derive equations (8) and (9).

Prob. 10-2. Write the differential equations for the circuits of Figs. 1, 2, 3, and 4 using the p-notation.

Prob. 11-2. Sketch a mechanical analogue of Fig. 4.

Prob. 12-2. The usual type of low-pass electric filter consists of a system of series inductances and shunt capacitance. Write the differential equations for such a filter of four sections terminated in a resistance R. Sketch a mechanical analogue such that when a sinusoidal translatory motion of sufficiently low frequency is applied to the input, a similar motion of nearly the same amplitude appears at the output.

Prob. 13-2. An oscillating torque which is periodic but not sinusoidal is applied to a shaft. The power is taken off a second shaft having the same center line as the first. Devise a mechanical filter, of three T-sections, to connect the two shafts and to eliminate all frequency components above 200 cycles/sec. The cut-off frequency is $1/(\pi \sqrt{LC})$.

Prob. 14-2. One type of band-pass filter has capacitances for shunt elements, while its series elements consist of capacitances and inductances in series. It is proposed to make a “hydraulic band-pass filter” to be used in a submarine signal system to eliminate sound waves of too high or too low a frequency. What might be its construction?

Prob. 15-2. An oscillograph vibrator is a highly-damped pendulum. The deflecting torque on the vibrator is proportional to the current through it. What are the parameters of the vibrator system? What is the electric-circuit analogue?
Prob. 16-2. A crystal, in the form of a thin layer the area of which is large compared to its thickness, is dropped into pure water and dissolved. Considering this as a problem of pure diffusion and neglecting effects near the edge of the crystal, what constants enter into the formulation of the problem, and what is the analogous electric circuit?

CHAPTER III
THE IMPEDANCE FUNCTION

To analyze for the behavior of a network means to solve the differential equations of the network; that is, to find expressions for the various currents which, substituted into these equations, will satisfy the initial and final conditions, and the terminal conditions if there are distributed parameters present.

We seek, then, expressions for the current which, substituted into the left-hand members, will give the right-hand members. To such expressions could evidently be added expressions which, substituted into the left-hand members, would give zero; and the sum of these expressions would still satisfy the equations. The solution of a set of linear differential equations thus falls into two parts: the first is the particular integral, and the second, the complementary function. The solution of a set of differential equations can also be divided into the steady-state term and the transient term: a part which is in form like the applied force, and a part which approaches zero as $t$ increases. Ordinarily these parts are obtained on solving for a particular integral and a complementary function. We deal at present with the steady-state term: that is, the series of current expressions which will give the impressed forces when substituted into the left-hand side of the equations.

2. Response to Exponential Electromotive Force. From a mathematical standpoint, the simplest form of electromotive force which can be applied to a network is one which varies exponentially. This is because of the simplicity of the derivative and the integral of an exponential.

If a current of the form:

$$i = I e^{at}$$
flows through a resistance $R$, the voltage resulting is

$$e = Ri.$$  

If it flows through an inductance $L$, the voltage is

$$e = Lp'i = LpIe^b = Lbi.$$  

If it flows through a capacitance $C$, the voltage is

$$e = \frac{1}{Cp}i = \frac{1}{Cp}Ie^b = \frac{1}{Cb}i.$$  

When an exponential electromotive force is applied to a linear network, the expression for the resulting current in each branch will contain an exponential steady-state term of the same form. It must evidently be of this form if it is to yield an expression equal at all times to the applied e.m.f., for the form of an exponential does not change when differentiated or integrated. If an electromotive force,

$$e = E_1e^{bt}$$  

be applied to Circuit 1 of a network, the resulting steady-state currents may be written

$$i_1 = I_1e^{bt}$$

$$i_2 = I_2e^{bt}$$

etc.

where $I_1$, $I_2$, etc. are constants we wish to evaluate. The value of $b$ is assumed here to be such that the determinant, to be formed below in (26), will not be zero.

Substitute these expressions for the currents into the general equations and write $z(b)$ for $R + Lb + \frac{1}{Cb}$, obtaining

$$z_{11}(b)I_1e^{bt} + z_{12}(b)I_2e^{bt} + \cdots + z_{1n}(b)I_ne^{bt} = E_1e^{bt}$$

$$z_{21}(b)I_1e^{bt} + z_{22}(b)I_2e^{bt} + \cdots + z_{2n}(b)I_ne^{bt} = 0$$

$$\vdots$$

$$z_{n1}(b)I_1e^{bt} + z_{n2}(b)I_2e^{bt} + \cdots + z_{nn}(b)I_ne^{bt} = 0$$

If the exponential which occurs in each term is eliminated,

we have left simply a set of algebraic equations containing the values of $I$:

$$z_{11}(b)I_1 + z_{12}(b)I_2 + \cdots + z_{1n}(b)I_n = E_1$$

$$z_{21}(b)I_1 + z_{22}(b)I_2 + \cdots + z_{2n}(b)I_n = 0$$

$$\vdots$$

$$z_{n1}(b)I_1 + z_{n2}(b)I_2 + \cdots + z_{nn}(b)I_n = 0.$$  

These equations may be solved to determine the values of $I$ by the usual methods applicable to first-degree algebraic equations, giving

$$I_k = \frac{M_{1k}(b)}{D(b)} E_1$$  

(23)

where $D(b)$ is the determinant:

$$\begin{vmatrix}
  z_{11}(b) & z_{12}(b) & \cdots & z_{1n}(b) \\
  z_{21}(b) & z_{22}(b) & \cdots & z_{2n}(b) \\
  \vdots & \vdots & \ddots & \vdots \\
  z_{n1}(b) & z_{n2}(b) & \cdots & z_{nn}(b)
\end{vmatrix}$$

(24)

and $M_{1k}$ is the minor (together with its proper sign) of the first row and $k$th column. For example,

$$I_2 = \frac{M_{12}(b)}{D(b)} E_1$$  

(25)

where $M_{12}(b)$ is

$$\begin{vmatrix}
  z_{21}(b) & z_{22}(b) & \cdots & z_{2n}(b) \\
  z_{31}(b) & z_{32}(b) & \cdots & z_{3n}(b) \\
  \vdots & \vdots & \ddots & \vdots \\
  z_{n1}(b) & z_{n2}(b) & \cdots & z_{nn}(b)
\end{vmatrix} (-1)^{1+2}$$

(26)

The abbreviation:

$$Z_{1k}(b) = \frac{D(b)}{M_{1k}(b)}$$

will be used, so that

$$I_k = \frac{E_1}{Z_{1k}(b)}$$

(28)

3. Alternating-Current Steady-State Solution. By means of an artifice, commonly used but not always stated, it is
equally easy to solve for the steady-state response to an alternating applied e.m.f. This is the **symbolic method** of solving alternating-current circuits.

Consider that in Circuit 1 there is applied the electromotive force,

\[ e = E_1 \cos \omega t. \]

This is identical with\(^1\)

\[ e = \frac{E_1}{2} e^{j\omega t} + \frac{E_1}{2} e^{-j\omega t}. \]

The applied e.m.f. can now be considered in two parts; and, the circuit being linear, they can be applied separately and the results added. Since each term is exponential, the results of the preceding section may be used directly, with \(j\omega\) in place of \(b\).

The loop current in Circuit \(k\) due to the first term is

\[ \frac{E_1}{2 Z_{1k}(j\omega)} e^{j\omega t} \]  \(\text{(29)}\)

and due to the second term,

\[ \frac{E_1}{2 Z_{1k}(-j\omega)} e^{-j\omega t}. \]  \(\text{(30)}\)

An examination of the determinant shows that \(Z(j\omega)\) and \(Z(-j\omega)\) are conjugate complex quantities. Thus,

\[ Z_{1k}(j\omega) = \frac{1}{Z_{1k}(j\omega)} \frac{1}{Z_{1k}(j\omega)} e^{j\phi} = Z_{1k}(j\omega) e^{j\phi} \]  \(\text{(31)}\)

\[ Z_{1k}(-j\omega) = \frac{1}{Z_{1k}(j\omega)} \frac{1}{Z_{1k}(j\omega)} e^{-j\phi} = Z_{1k}(j\omega) e^{-j\phi} \]

so the total loop current in Circuit \(k\) is

\[ i_k = \frac{E_1}{2 |Z_{1k}(j\omega)|} (e^{j\omega t} e^{-j\phi} + e^{-j\omega t} e^{j\phi}) \]

\[ = \frac{E_1}{|Z_{1k}(j\omega)|} \cos (\omega t - \phi). \]  \(\text{(32)}\)

\(^1\) We here use \(j = \sqrt{-1}\), since in electrical problems it is desirable to reserve \(i\) for current. In mathematical texts, however, \(i\) is usually used for the pure imaginary.

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**IMPEDANCE FUNCTION**

We have here carried through the entire process. Usually it is abbreviated and only one of the terms used, a convention being introduced so that the \(j\) is removed from the final expression. When both terms are used it disappears automatically as it necessarily must, for the currents that flow are real, not imaginary. Many a student of electrical engineering has wondered what is meant by an imaginary expression for a real current.

The \(Z(j\omega)\) above is the alternating-current impedance of the network. It may be formed from the determinants as above, or often more readily by combining the impedances of elements in series and parallel in accordance with methods very familiar to electrical engineers.

\(Z_{1k}(j\omega)\) is the impedance of the network as examined from a point in Circuit 1. When divided into the e.m.f. applied to this circuit, it will give the loop current which flows in this circuit; its magnitude giving the magnitude of current and its angle giving the phase angle between current and voltage.

\(Z_{1k}(j\omega)\) is a transfer impedance. Divided into an electromotive force applied in Circuit 1, it gives the loop current, which will flow in Circuit \(k\).

\[ Z_{1k}(j\omega) = Z_{1k}(j\omega). \]

From this follows the **reciprocal theorem**: If a given e.m.f. in one branch produces a certain current in a second branch, then with the conditions reversed and the given e.m.f. applied in the second branch, the same current as found before in the second branch will now be found in the first branch. In other words, if the applied e.m.f. remains unchanged, the source of e.m.f. and the current-measuring instrument may be interchanged in a network without altering the current reading. This theorem will be extended below.

4. **Importance of the Impedance Function.** The impedance function \(Z\) of a network completely specifies it electrically, provided the network has fixed parameters. If there are only two terminals of the network available, then the \(Z\)
at this point contains all possible information concerning the network. If there are two pairs of terminals available, the Z at each point and the transfer impedance between them gives the information.

Evidently, in accordance with the preceding section, if two networks have the same Z-function at some point, they will respond identically to the same alternating voltage applied at this point, no matter what may be their internal connections. A similar statement may be readily made when several pairs of terminals are available and all the Z's are identical. It is not so evident, but is just as true as we shall see later, that the two networks will respond in identical manner\(^1\) to a voltage of any form whatever applied to their terminals. In other words, two networks which have the same Z at available terminals cannot be distinguished by any electrical tests whatever so long as all measurements are confined to conditions at these terminals; assuming, of course, that the applied voltages are kept within bounds so that all networks remain fixed. As an example, consider the three very dissimilar networks shown in Fig. 9. The Z-function of each is simply equal to R. If all three were in boxes with only the terminals a-b available, it would be impossible to tell them apart by electrical tests only.\(^2\)

\(^1\) With identical transient currents as well as identical steady-state currents.


It is evident that the impedance function Z is an important characteristic of a network. It is readily formed, and it contains within itself all the information necessary for the determination of the circuit performance under any conditions whatever. Unfortunately, the relationships by which it will yield this information are not always simple.

5. Impedance of Distributed Circuit. So far only circuits with lumped parameters have been treated. The impedance function can also be readily obtained for circuits with distributed parameters. With distributed circuits, a partial differential equation takes the place of a set of simultaneous total differential equations. In order to find the impedance function, we find the ratio between the steady-state voltage and current for a sinusoidal, or simpler still, for an exponential voltage. We shall do so for the most usual type of distributed circuit, as represented by the electric transmission line. The same treatment applies to hydraulic or acoustic pipe lines or to mechanical vibrating systems in which the parameters are distributed.\(^1\)

In the line of Fig. 10 let the parameters be \(r\) ohms, \(l\) henries, \(c\) farads, each per unit length of line. There may be also leakage between lines given by a conductance of \(g\) mhos per unit length.

Consider a differential length \(dx\) of line. The potentials at the two ends of this element of length differ by an amount given by the ohmic and inductive drop in the element, or

\[
\frac{\partial e}{\partial x} \, dx = -ri \, dx - l \, dx \frac{\partial i}{\partial t}
\]

(33)

where the minus signs are due to the convention that the positive direction of both \(i\) and \(x\) is to the right. In this case, for instance, Maxfield and Harrison, loc. cit.
expression, \( e \) and \( i \) mean the voltage and current at a point \( x \) units from the generator end. Similarly, the currents at the two ends differ by reason of current which leaks between lines and current which flows into the element capacitance. Thus,

\[
\frac{\partial i}{\partial x} = -ge \, dx - c \, dx \frac{\partial e}{\partial i}.
\]  
(34)

These two equations:

\[
\begin{align*}
\frac{\partial e}{\partial x} &= -ri - i \frac{\partial i}{\partial t} \\
\frac{\partial i}{\partial x} &= -ge - c \frac{\partial e}{\partial t}
\end{align*}
\]  
(35)

are the partial differential equations of the system.

If currents are now considered to be of the exponential form, then due to the properties of the exponential, steady-state voltages will be of the same exponential form, or

\[
\begin{align*}
i &= I e^{a}
\end{align*}
\]  
(36)

\[
\begin{align*}
e &= E e^{a}
\end{align*}
\]  
where \( I \) and \( E \) are functions of \( x \) alone.

Our equations then become total differential equations; the time disappears when the exponential is eliminated.

\[
\begin{align*}
\frac{dE}{dx} &= -rI - lbI \\
\frac{dI}{dx} &= -qE - cbE
\end{align*}
\]  
(37)

Eliminating between these,

\[
\begin{align*}
\frac{d^2 E}{dx^2} &= (r + lb) (g + cb) E \\
\frac{d^2 I}{dx^2} &= (r + lb) (g + cb) I
\end{align*}
\]  
(38)

Abbreviating with

\[
(r + lb) (g + cb) = \beta^2
\]  
(39)

we have simply

\[
\begin{align*}
\frac{d^2 E}{dx^2} &= \beta^2 E \\
\frac{d^2 I}{dx^2} &= \beta^2 I
\end{align*}
\]  
(40)

Solutions of these are

\[
\begin{align*}
E &= A e^{ax} + B e^{-bx} \\
I &= C e^{ax} + D e^{-ax}
\end{align*}
\]  
(41)

By noting that

\[
\frac{dI}{dx} = -(g + cb) E
\]  
(42)

for all values of \( x \), we can reduce the constants of integration to two, and write

\[
\begin{align*}
E &= A e^{ax} + B e^{-bx} \\
I &= -A \frac{g + cb}{\beta} e^{ax} + B \frac{g + cb}{\beta} e^{-bx}
\end{align*}
\]  
(43)

With the further abbreviation:

\[
z_0 = \sqrt{\frac{r + lb}{g + cb}}
\]  
(44)

these are simply

\[
\begin{align*}
E &= A e^{ax} + B e^{-bx} \\
I &= -A \frac{e^{ax}}{z_0} + B \frac{e^{-bx}}{z_0}
\end{align*}
\]  
(45)

Now the line may be terminated in various ways at its ends. As a special case, suppose that it is supplied with voltage at one end and extends to an infinite distance. It is evident physically that if \( x \) increases without limit, the voltage and current cannot rise without limit. Hence \( A \) in the above must be zero.

\[
\begin{align*}
E &= B e^{-bx} \\
I &= B \frac{e^{-bx}}{z_0}
\end{align*}
\]  
(46)
The ratio of voltage to current at any point of the line is then simply equal to \( z_0 \). The impedance function for a long line of this sort, examined at any point, is then

\[
Z(b) = \sqrt{\frac{r + lb}{g + cb}}. \tag{47}
\]

The alternating-current impedance is

\[
Z(j\omega) = \sqrt{\frac{r + j\omega}{g + j\omega}}. \tag{48}
\]

The impedance function for various other terminations of the line is similarly obtained and will be found in texts on transmission lines.\(^1\) We need review only this one case to illustrate the treatment of the partial differential equation.

6. **Steady-state Analysis.** The general problem of the steady-state response of linear networks to alternating impressed electromotive forces is of great importance to engineers, but cannot be entered into here. This text deals rather with the transient behavior of networks, and more particularly with the analysis of circuit transients by operational processes, especially those of Heaviside. Our interest in the impedance function comes from the fact that it is the most convenient way of summing up the electrical characteristics of the network.

There is a considerable literature on the steady-state performance of networks. For the distributed circuits of power transmission, the usual vector diagrams which are so useful in steady-state analysis have been developed into circle diagrams which summarize an enormous amount of information in small compass.\(^2\) Networks in which the same circuit is repeated several times and which exhibit strong frequency selective properties are called filters; and their steady-state treatment is found in a large number of articles and texts.\(^1\) Certain special networks, such as the transformer, have treatments especially adapted to them. In all these cases the treatment can be applied to non-electrical networks by analogy.\(^2\)

**PROBLEMS ON CHAPTER III**

**Prob. 1-3.** Solve for the steady-state currents in both inductances of Fig. 1, using determinants and minors. Consider the impressed voltage to be \( E = j\omega L \).

**Prob. 2-3.** A network of three circuits has all three parameters in each circuit. Circuits 1 and 2 are coupled by a common condenser, while Circuits 2 and 3 are coupled by a common resistance. Find the impedance function for a point in Circuit 1: (a) by combining impedances in series and parallel, (b) by using determinants.

Repeat (a) and (b) for the transfer impedance \( Z_{12} \).

**Prob. 3-3.** A long copper rod is to be heated at one end, and the flow of heat studied. Set up the differential equations, and form the impedance function. (This will be discussed in Chapter XI, but may be considered here by analogy with the long transmission line, noting what parameters are negligible.)

**Prob. 4-3.** A thin-walled rubber tube is filled with water, closed at one end, and connected to a reciprocating pump at the other end. Note the parameters of the tube, and its impedance function. A somewhat similar circuit appears in considering the flow of blood in arteries.

\(^1\) Cf. G. W. Pierce, Electric Oscillations and Electric Waves, McGraw-Hill, 1920, Chapter XVI.


\(^6\) Johnson and Shea, Mutual Inductance in Wave Filters, B.S.T.J., 4, 1925, p. 52.

\(^2\) Notably in the case of acoustic filters. Consult the papers of G. W. Stewart, loc. cit. or that of W. P. Mason, loc. cit.
Prob. 5-3. Two power transmission lines built in identical manner run out 20 and 50 miles respectively from a power station. They are connected to the same bus bars at the station; and terminate at the distant end in impedances \( Z_1 \) and \( Z_2 \) respectively. Set up the expression for the impedance function of each line alone, and for the total impedance measured at the bus bars. Formulate the transfer impedance for one line between bus bars and load. See texts on transmission lines (e.g., Woodruff, Principles of Electric Power Transmission and Distribution).

Prob. 6-3. The impedance function of a lumped network may be readily found by the usual rules for combining the alternating-current impedances of the elements in series and parallel. This will always work except in the case where a portion of the network appears in a form similar to a Wheatstone bridge. Then a delta-star transformation of three elements will always eliminate the difficulty. This matter should be reviewed in any good text on alternating-current circuits (e.g., Lawrence, Principles of Alternating Currents). Satisfy yourself that this procedure will yield both driving-point and transfer impedance functions by setting up various lumped networks and outlining the procedure of making the combinations. Transformers should first be replaced by their equivalent circuits, and the same with electrostatic couplings when such couplings are not simply made up of condensers.

Prob. 7-3. Determine the velocity in the mechanical resistance of Prob. 12-2 when the applied force is \( E \cos \omega t \) dynes. What is the transfer impedance between the first and last circuits?

Prob. 8-3. Solve for the steady state of \( i \) in the analogous electric circuit of Prob. 8-2, using an applied force of 100 \( \sin 0 \pi t \).

Prob. 9-3. A loud speaker having an inductance of 1 h. and resistance of 1000 ohms is connected in series with a capacitance of 2 mfd. and the combination is shunted by a 30-henry choke of 500-ohms resistance. To this parallel group is connected an alternating voltage, \( E \sin \omega t \), through a 2000-ohm resistance. In this amplifier-output device the 2000 ohms and the voltage \( E \sin \omega t \) represent approximately the plate circuit of the last triode of an amplifier. Plot the velocity of the speaker diaphragm as a function of frequency between 50 and 5000 cycles per sec. with constant alternating voltage, \( E \sin \omega t \). In what frequency range would you consider this output device satisfactory? Assume that the diaphragm displacement is at all times directly proportional to the current through the loud-speaker winding, calling the velocity unity at 1000 cycles/sec.

\[ \text{Fig. 11. The Unit Function.} \]

\[ \text{Fig. 12. Discontinuous Exponential Function.} \]

CHAPTER IV

THE INDICIAL ADMITTANCE

The transient analysis of networks can be best based upon the response to a suddenly-impressed unit voltage. Such a voltage may be impressed on a network by closing a switch and thus connecting a battery of unit e.m.f., or it may be impressed by suddenly inserting into the closed network a source of one volt. In either case the voltage to which the network is subjected is zero up to a certain instant, and unity thereafter. This fundamental sort of impressed voltage will be called a unit voltage.

More generally, we shall specify by unit function that discontinuous function\(^1\) of the time which is zero until \( t \) equals zero and unity thereafter. It is plotted in Fig. 11. Following Heaviside, we shall denote such a function by the symbol \( I \). This unusual way of writing a time function is convenient because it can also be used in connection with other functions as a brief way of denoting discontinuity. Thus \( e^{at} \) represents the usual exponential function of the time, but \( e^{at} I \) represents that function of time shown in Fig. 12.

\(^1\) In mathematical texts this function is often written \( \frac{1}{2} (1 + \text{sgn } t) \).
which is zero until \( t \) equals zero and exponential thereafter. The expression is the product of two time functions — an exponential, and a unit function — but the \( t \) can also be regarded as a symbol calling attention to the fact that the quantity to which it is appended has a discontinuity at the origin.

The unit function is important, for it is the simplest possible discontinuous function. The transient analysis of networks is the study of their behavior when subjected to discontinuities. The simplest discontinuity has a fundamental significance, for all transient behavior can be expressed in terms of it.

2. Definition of Indicial Admittance. The response of a network to unit e.m.f. \( (1) \) is called the indical admittance of the network, and is denoted by \( A(t) \).\(^1\) \( A(t) \) is thus the current which flows in the network when unit e.m.f. is suddenly applied, it being assumed throughout that the network is initially at rest with zero current and voltage everywhere. \( A(t) \) thus gives the fundamental direct-current transient of the network.

The current that will flow in Circuit \( k \) when unit electromotive force is applied to Circuit \( h \) is denoted by

\[
A_{kh}(t)
\]

and is called the transfer indical admittance between Circuits \( h \) and \( k \).

The indical admittance of a network completely characterizes the network. This same statement was made in regard to the impedance function. There must, therefore, be a definite relation between the two; so that if \( Z \) for a network is known, \( A \) for the same network can be found,

\(^1\) The term "indical admittance" was introduced by J. R. Carson, and has been generally adopted. For graphs of this function for a number of different circuits, see Carson, Trans. A.I.E.E., 1919, or Carson and Zobel, Transient Oscillations in Electric Wave-Filters, B.S.T.J., July, 1923, p. 1.

and vice versa. In fact there are several such relations which are mathematically identical when used under conditions to which they all apply, for they express the same interdependence but in widely different forms. A considerable portion of this text is devoted to their development and use.

3. The Steady-State Solution. The indical admittance of a network can of course be found by classic methods; that is, the differential equations can be solved and the constants evaluated by means of the initial conditions which correspond to the application of the unit electromotive force.

When unit voltage is applied to Circuit \( h \), the current in Circuit \( k \) will be given by a steady-state term and a transient term. The steady-state term for \( i_k \) is simply

\[
\frac{1}{Z_{kh}(0)}
\]

This can be taken from our previous solution for the steady-state response to an applied voltage of the exponential form by putting \( b = 0 \), which reduces the exponential to a unit voltage.

The value of \( Z(0) \) for a network — that is, the impedance of the network at zero frequency — is, of course, nothing but the direct-current resistance. This can be found most readily by inspection, ignoring all inductances and condensers in the circuit and combining the individual resistances in series and in parallel. In this process the inductances are considered of zero impedance and the condensers of infinite impedance.

4. Classic Treatment of Transient Solution. In order to obtain the transient solution, we need to find the complementary function which satisfies the equations:

\[
\begin{align*}
\varepsilon_1(p)i_1 + \varepsilon_2(p)i_2 + \cdots + \varepsilon_m(p)i_m &= 0 \\
\varepsilon_2(p)i_1 + \varepsilon_2(p)i_2 + \cdots + \varepsilon_m(p)i_m &= 0 \\
\cdots & \\
\varepsilon_m(p)i_1 + \varepsilon_m(p)i_2 + \cdots + \varepsilon_m(p)i_m &= 0
\end{align*}
\] (49)
Let us assume that a solution of these equations exists in the form:

$$i_1 = K_{11}e^{\lambda t},$$
$$i_2 = K_{22}e^{\lambda t}, \text{ etc.}$$ \hspace{1cm} (50)

If we substitute these into (49) and divide out the exponential, the following algebraic relations are obtained:

$$z_{11}(\lambda)K_{11} + z_{12}(\lambda)K_{22} + \cdots + z_{n1}(\lambda)K_{na} = 0$$
$$z_{21}(\lambda)K_{11} + z_{22}(\lambda)K_{22} + \cdots + z_{2n}(\lambda)K_{na} = 0$$
$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$
$$z_{n1}(\lambda)K_{11} + z_{n2}(\lambda)K_{22} + \cdots + z_{nn}(\lambda)K_{na} = 0.$$ \hspace{1cm} (51)

Such a set of algebraic equations has a solution only when the determinant of the system vanishes; that is, when

$$D(\lambda) = 0.$$

$D$ has the same meaning as in Chapter III. This is an algebraic equation in $\lambda$. In general it will have $2n$ roots:

$$\lambda_1, \lambda_2, \lambda_3, \cdots \lambda_{2n}$$

where $n$ is the number of circuits in the network, although it may have less than this number if some parameters are absent. The equation $D(\lambda) = 0$ is called the determinantal equation of the network.

Expressions of the form:

$$i_k = K_{k1}e^{\lambda t} + K_{k2}e^{\lambda t} + \cdots (2n \text{ terms})$$ \hspace{1cm} (52)

therefore satisfy our differential equations, as may be verified by substitution. Thus we have the form of the transient portion of our solution, although the values of the constants of integration $K_{k1}, K_{k2}, \cdots$ are as yet undetermined.\(^1\)

\(^1\) Note that the double subscript used here with $K$ identifies the particular current and the particular root to which that value of $K$ belongs.

The expressions of the form $K e^{\lambda t}$ are called the normal modes of the network, and the values $\lambda_1, \lambda_2, \cdots$ are called the generalized natural angular velocities, for reasons that will appear.

5. **Determination of Generalized Natural Angular Velocities.** Since we have

$$Z(\lambda) = \frac{D(\lambda)}{M(\lambda)}$$ \hspace{1cm} (53)

we shall in general obtain the same roots, or the same values of $\lambda_1, \lambda_2, \cdots$, if we equate $Z(\lambda)$ to zero as when we equate $D(\lambda)$ to zero. For the present it shall be assumed that all the roots are different. The case of coincident roots will be discussed later.\(^1\) We shall also scrutinize cases in which a different set of roots will be obtained from $Z$ than from $D$.

One way of finding the natural angular velocities of a network is hence to form the alternating-current impedance of the network:

$$Z(j\omega)$$

in any convenient manner, replace $j\omega$ by $\lambda$, and solve

$$Z(\lambda) = 0.$$\(^2\)

Usually in a transient solution a term will appear corresponding to each value of $\lambda$ obtained as a root. There are special conditions under which some of these terms may not appear. This will be probed in Chapter VII.

In general the same values of $\lambda$ will be obtained on equating $Z(\lambda)$ to zero, no matter which point in the network is considered in forming $Z$. Evidently we shall always get the same roots as from $D(\lambda)$ unless one of these roots happens also to make $M(\lambda)$ zero. This special case also requires some discussion later. For the present, however, it will be assumed that $Z$ may be formed looking into the network from any convenient point. This corresponds to the fact that in general the same modes of natural oscillation will

\(^2\) See p. 91.
appear in a network, no matter how it is jarred out of equilibrium. The exceptions to this rule require special treatment.

6. Form of Normal Modes. The solution of an \( n \)th degree algebraic equation offers much difficulty when \( n \) is large, especially when some of the roots are complex. Certain vector methods of solution have been developed which are helpful. When the resistances in a circuit are small, approximate methods can be employed.\(^1\)

The roots obtained are either real or in conjugate complex pairs. If real, they are negative. A positive root would denote a current increasing without limit. This is not possible in a dissipative network. Sometimes in dealing with an infinite distributed circuit, spurious positive roots may appear. These are considered in Chapter XIII. Complex roots will ordinarily have a negative real portion. A pair of conjugate roots may appear thus:

\[
\begin{align*}
\lambda_1 &= -\alpha_1 + j\omega_1 \\
\lambda_2 &= -\alpha_1 - j\omega_1
\end{align*}
\]  

(54)

The corresponding terms of the complementary function are then

\[
K_{k1}e^{-\alpha_1 t} + K_{k2}e^{-\alpha_2 t}
\]  

(55)

or

\[
e^{-\alpha t}(K_{k1}e^{j\omega t} + K_{k2}e^{-j\omega t})
\]  

(56)

or

\[
e^{-\alpha t}[(K_{k1} + K_{k2}) \cos \omega t + j(K_{k1} - K_{k2}) \sin \omega t].
\]

(57)

In any specific solution the constants of integration, \( K_{k1} \) and \( K_{k2} \), are also bound to come out conjugate complex quantities, so that the expression may be reduced to

\[
Be^{-\alpha t} \sin (\omega t + \phi)
\]  

(58)

where \( B \) and \( \phi \) are real constants of integration. This must occur, for otherwise we should have a complex expression for a real current — a result which is inadmissible in the absence of a convention by which the imaginary terms may be interpreted into corresponding real terms. Such a convention is often used in the symbolic solution of networks in the steady state, but it is not present here in this transient study.

The quantity \( \alpha_1 \) is called a natural decrement of the network, and \( \omega_1 \) is called a natural angular velocity of the network. In an oscillatory network there will in general be

as many natural angular velocities as there are circuits in the network. \( \omega_1/2\pi \) is a natural frequency of the network. To each natural frequency corresponds a particular damping factor \( e^{-\alpha t} \).

The combination of two complex roots gives a damped sinusoid, and the transient behavior of an oscillatory network is given as a sum of damped sinusoids — in general, as many as there are circuits. If there are real roots they can be similarly combined in pairs, giving a surge as in Fig. 13. Single, unpaired, real roots may also appear.
7. The Amplitudes of Oscillation. The indicial admittance of a network has now been completely determined except for the amplitudes $K$ of the transient terms. The unit voltage was applied in Circuit $h$, and the transfer indicial admittance between Circuits $h$ and $k$ was desired. It has the form:

$$A_{hk}(t) = \frac{1}{Z_{hk}(0)} + K_{h1}e^{st} + K_{h2}e^{2st} + \cdots (2n \text{ terms}).$$  

(59)

So far the classic theory has proceeded smoothly; but when an attempt is made to determine the values of $K$, the amplitudes of oscillation, the classic method encounters insuperable difficulties except in very simple cases. The primary reason for the relative success of operational methods of circuit analysis lies in the fact that it avoids, to a certain extent at least, the difficulty of determining these constants of integration.

Since there are $2n$ values of $K$, we need $2n$ equations to determine them. One of these equations is readily written, and it will pay us to write it even though we cannot proceed to set up the others for the general case.

To find the initial conditions in the network — that is, the distribution of current at the instant just after the switch is closed — we can simplify the network by removing all condensers by short-circuiting them, and all inductances by cutting them out of circuit. This is exactly the reverse process of that for finding the final steady-state current. It follows because a condenser can support no voltage until a finite time has elapsed during which it can accumulate a charge, and an inductance can carry no finite current until the current has had a finite time to build up. It will be remembered that all condensers and inductances are considered to be initially without charge or current. The same effect of simplifying the circuit will be obtained if its response to an infinite frequency is considered, for condensers will then support no voltage and inductances will carry no current. The initial current in the circuit will then be given by

$$\frac{1}{Z_{hk}(\infty)}.$$  

From this, inserting $t = 0$ in our expression, we can obtain one equation for the $K$’s:

$$K_{h1} + K_{h2} + \cdots = \frac{1}{Z_{hk}(\infty)} - \frac{1}{Z_{hk}(0)}.$$  

(60)

The right-hand side of this equation is the difference between the initial and final currents. In a similar manner an equation can be written for the $K$’s of any other branch, but they are different $K$’s of course. By writing the relation between the current in one branch and that in another it is possible to connect them with the ones we wish to determine. This process is altogether too complex to be carried through in general. We shall content ourselves with a simple illustration.

8. An Example. Consider the transformer of Fig. 14. We seek the indicial admittance of the network from a point in one circuit. The differential equations are now

$$\begin{align*}
(R_1 + L_1)p_{i1} + L_1q_{i2} &= E, \\
L_2q_{i1} + (R_2 + L_2)p_{i2} &= 0.
\end{align*}$$

(61)

The determinantal equation is

$$\begin{vmatrix}
R_1 + L_1 \lambda & L_1 \lambda \\
L_1 \lambda & R_2 + L_2 \lambda
\end{vmatrix} = 0$$

or, since $L_{21} = L_{12},$

$$(R_1 + L_1 \lambda) (R_2 + L_2 \lambda) - L_1^2 \lambda^2 = 0$$

(63)

from which can be obtained two values of $\lambda$:

$$\lambda_1, \lambda_2 = -\frac{R_1L_2 + R_2L_1}{2(L_1L_2 - L_1^2)} \pm \frac{1}{2} \sqrt{\frac{(R_1L_2 + R_2L_1)^2}{(L_1L_2 - L_1^2)^2} - \frac{4R_1R_2}{L_1L_2 - L_1^2}}.$$  

(64)
Notice that \( L_{12} \), the mutual inductance, is less than, or at most equal to, the geometric mean of the self-inductances \( \sqrt{L_1L_2} \). Thus \( \lambda_1 \) and \( \lambda_2 \) are always negative.

The impedance \( Z_{ii}(p) \) is
\[
Z_{ii}(p) = \frac{D(p)}{M_{ii}(p)} = \frac{(R_1 + L_1 p)(R_2 + L_2 p) - L_{12} p^2}{R_2 + L_2 p} = R_1 + L_1 p - \frac{L_{12} p^2}{R_2 + L_2 p}.
\]

(65)

This corresponds to the well-known expression for the alternating-current impedance of a transformer:
\[
Z_{ii}(j\omega) = z_1(j\omega) + \frac{L_{12} \omega^2}{2z_2(j\omega)}
\]

(66)

where \( z_1 \) and \( z_2 \) are respectively the impedances of primary and secondary alone. Now, inserting \( p = 0 \),
\[
\frac{1}{Z_{ii}(0)} = \frac{1}{R_1}
\]

(67)

for the final steady state.

Also, inserting \( p = \infty \), we have
\[
\frac{1}{Z_{ii}(\infty)} = 0
\]

(68)

for the initial current. In this simple example, these values are of course apparent from inspection. The indicial admittance is then
\[
A_{ii}(t) = \frac{1}{R_1} + K_{11} e^{\lambda_1 t} + K_{12} e^{\lambda_2 t}
\]

(69)

and we have one equation to determine the \( K \)'s:
\[
K_{11} + K_{12} = -\frac{1}{R_1}.
\]

(70)

Now the current in the secondary is
\[
A_{12}(t) = K_{21} e^{\lambda_1 t} + K_{22} e^{\lambda_2 t}.
\]

(71)

But an expression for this current can also be obtained by noting that, for each exponential term of the primary such as \( K_{11} e^{\lambda_1 t} \), there is an induced secondary voltage \( L_{12} \lambda_1 K_{11} e^{\lambda_1 t} \), and a term of secondary current:
\[
\frac{L_{21} \lambda_1 K_{11}}{R_2 + L_2 \lambda_1} e^{\lambda_1 t}.
\]

Thus the secondary current can be written as
\[
A_{12}(t) = \frac{L_{21} \lambda_1 K_{11}}{R_2 + L_2 \lambda_1} e^{\lambda_1 t} + \frac{L_{21} \lambda_2 K_{12}}{R_2 + L_2 \lambda_2} e^{\lambda_2 t}.
\]

(72)

Now, setting this current equal to zero for the initial instant, \( t = 0 \), we have
\[
\frac{\lambda_1}{R_2 + L_2 \lambda_1} K_{11} + \frac{\lambda_2}{R_2 + L_2 \lambda_2} K_{12} = 0.
\]

(73)

There are now two equations from which \( K_{11} \) and \( K_{12} \) can be determined:
\[
K_{11} = \frac{1}{\lambda_1 \lambda_2 - \frac{L_1}{R_2 + L_1 \lambda_1} - 1}
\]

\[
K_{12} = \frac{1}{\lambda_1 \lambda_2 - \frac{L_1}{R_2 + L_1 \lambda_1} - 1}
\]

(74)

and the expression for the indicial admittance is completely determined.

9. Various Initial Conditions. Thus the classic method for the determination of the amplitudes of the modes of free oscillation drives us to special maneuvers in particular problems and gives no general method of procedure. There are many of these devices. We cannot go into them in detail here, but a few can be mentioned.

We have already seen how the network can be simplified to obtain initial and final currents. We have also seen how a current in one circuit can be transformed to find the current in another circuit by computing voltages and currents.
for each exponential term that appears. There are also several other ways in which equations can be arrived at for determining the amplitudes, $K$. For example, the initial rate of increase of the current in a network is given by a consideration of the inductances alone. At the first instant, resistances which are in series with inductance support no voltage, for they have no current through them. The same is true for uncharged condensers which are in series with other parameters. The entire voltage is hence brought to bear on the inductances, and the rate of increase of current is given by the applied voltage divided by an equivalent inductance of the network. This equivalent inductance is obtained by combining inductances, ignoring everything else; or by setting all $R$'s equal to zero, and all $C$'s equal to infinity in the expression for the impedance. Thus in the example above, the impedance due to inductance alone is

$$L_1 = \frac{L_1}{L_2}$$

or the equivalent inductance is

$$L_1 = \frac{L_1}{L_2}$$

The initial rate of current increase is thus:

$$\frac{1}{L_1} = \frac{L_1}{L_2} = \lambda_1 K_{11} + \lambda_2 K_{12}$$  \hspace{1cm} (75)

obtained by taking the derivative of the current expression and then setting $t = 0$. This equation might have been used to determine the $K$'s in place of the one actually employed, and would have given, of course, the same results.

Other initial conditions are often useful. The initial rate of change of voltage across a condenser is the initial current divided by the capacitance. The second derivative of current in a circuit, such as the above, is given by the rate of change of the voltage applied to the inductance. This is best illustrated in a simple series circuit as in Fig. 15. Here, when $t = 0$,

$$\begin{align*}
i &= 0 & e_L &= E \\
p &= \frac{E}{L} & p e_L &= -\frac{RE}{L} \\
\dot{p} &= -\frac{RE}{L^2} & e_R &= 0 \\
e &= 0 & p e_R &= \frac{RE}{L} \\
p^2 &= \frac{E}{LC} & 
\end{align*}$$

and so on. There are hence always plenty of ways of getting at equations for the $K$'s in any specific problem. Unfortunately the procedure gets very complicated when there are several circuits in the network. In succeeding chapters we shall discuss general methods of arriving at the indicial admittance of networks, which will usually be found to work with less complication and difficulty. This is the main problem of the operational calculus — the ready determination of the indicial admittance.

We have discussed in this section only networks with lumped parameters. The classic treatment of the indicial admittance of circuits with distributed parameters is so involved that we shall not review it. It is in this analysis of distributed networks that operational processes become essential if we are to proceed in a practical way.

In all the above the network has been assumed initially in equilibrium; that is, with no current in the coils or voltage across the condensers. In other words it has been assumed that there is no initial storage of energy, either in electro-
magnetic or in electrostatic form. We shall discuss the treatment of networks with initial energy storage in Chapter XIV.

10. Impulsive Response. Before leaving the differential equations of classic treatment there is one point to be mentioned—that of impulsive response. We assumed in the above treatment that all three parameters were present in each circuit. In cases where some of them are missing there is no change of treatment, except for one instance which needs comment. This is the case when \( Z(\infty) \) comes out zero. The initial current is then infinite; and our treatment, in which we have assumed finite continuous expressions for the current, breaks down.

In order to meet this case, we must have across the points of voltage application a condenser unprotected by series inductance or resistance. Of course such an arrangement is a mathematical fiction, for a condenser cannot be built entirely devoid of inductance and resistance. The simple engineering way out of the difficulty is to insert in the expressions the series resistance of the condenser which is physically known to be present; and, no matter how small this resistance may be, the mathematical difficulties with infinite currents promptly disappear.

Another procedure that avoids trouble with infinite values is to solve for charges instead of currents, and obtain the current later from the derivative of charge, due regard being paid to discontinuities. When we suddenly apply unit e.m.f. to a condenser of unit capacitance, the current is mathematically infinite at \( t = 0 \) and zero thereafter. We shall find that this fundamental sort of impulsive response corresponds to a simple operational expression, the presence of which has sometimes caused perplexity. Yet if we reason this proposition in terms of charges, much of the mystery vanishes. At \( t = 0 \) when the switch is closed, a unit charge is (mathematically) instantly placed on the condenser, and that is all.

This case of impulsive response may occur in all the theorems which we shall derive on the basis of the differential equation formulation. It has been customary to make an exception to each rule or formula in order to bar such cases. We shall find, however, that by adding a special term, it is usually possible to include impulsive response in the regular treatment and avoid continual exceptions.

PROBLEMS ON CHAPTER IV

Prob. 1-4. For the circuit of Fig. 1, what is the determinantal equation? What are the normal modes, the natural decrements, the natural frequencies, and the damping factors?

Prob. 2-4. Sketch the following for positive and negative values of \( t \): \( t, e^{-t}, t^2, e^{-t}, te^{-t}, (1 - e^{-t})t + e^{-t} \).

Prob. 3-4. Sketch the following for positive and negative values of \( t \): \( t, e^{-t}, t^2, e^{-t}, te^{-t}, (1 - e^{-t})t + e^{-t} \).

Prob. 4-4. Obtain \( i_t \) for Fig. 2, using the following values: \( L_1 = 0 \), \( L_2 = 30 \text{ h} \), \( L_{12} = 10 \text{ h} \), \( R_1 = 10 \text{ ohms} \), \( R_2 = 15 \text{ ohms} \), and \( E = 230 \text{ volts} \). Close the switch at \( t = 0 \).

Prob. 5-4. Determine \( i_1 \) and \( i_2 \) in Fig. 1, using these values: \( L_1 = L_2 = 0 \), \( R_1 = 1 \text{ ohm} \), \( R_2 = 30 \text{ ohms} \), \( R_{12} = 10 \text{ ohms} \), \( C_1 = 50 \text{ mfd} \), \( C_2 = 10 \text{ mfd} \), and \( E = 6 \text{ volts} \). The switch is closed at \( t = 0 \). What is the indicial admittance of the network viewed from the battery?

Prob. 6-4. Find \( A_{11}(t) \) and \( A_{12}(t) \) for the coupled circuit of Fig. 14, adding a condenser of \( 5 \times 10^{-6} \) farads in series with the secondary. The parameters are \( R_1 = 10 \text{ ohms} \), \( R_2 = 5 \text{ ohms} \), \( L_1 = 0.5 \text{ h} \), \( L_2 = 1.0 \text{ h} \), \( L_{12} = 0.5 \text{ h} \). Take \( E = 0.1 \text{ volt} \). The switch is closed at \( t = 0 \).

Prob. 7-4. Find the velocity of the 300-gram pendulum bob in Prob. 8-2 as a function of time.