Multicommodity Flow

In settings like water pipes, water is water, and so is a single commodity.

What if water has colors, that need to be kept separate.

⇒ One source-destination pair cares about blue water, another source-destination pair cares about green water.

Multiple commodities that interact by sharing the same network with common capacity limitations.

⇒ Individual single commodity problems are not independent but have to be solved together.

Setting one easier is that each edge has capacity \( u_{ij} \) that restricts total flow of all commodities on that edge.

Let \( x_{ij}^k \) denote the flow of commodity \( k \) on edge \((i,j)\) and let \( x^k \) and \( c^k \) be the flow vector and per unit cost vector for commodity \( k \).

Then multicommodity flow problem for \( G=(V,E) \).

\[
\min \sum_{k=1}^{K} c^k x^k
\]

s.t. \( \sum_{k=1}^{K} x_{ij}^k \leq u_{ij} \) for all \((i,j) \in E\)  

\( \sum_{k=1}^{K} x^k \leq b^k \) for \( k=1,2,\ldots,K \)  

\( Nx^k = b^k \) for \( k=1,2,\ldots,K \) \( N \) mass balance constraint

where \( b(i)^k \) represents supply/amount of commodity \( k \) at node \( i \)

\( N \) is \( n \times m \) right, nonedge incidence matrix
Each column \( N_{ij} \) in matrix corresponds to variable \( x_{ij} \). Column \( N_{ij} \) has 1 in its row, -1 in its row, and zero elsewhere.

**Notes on model**

1. **Homogeneous goods assumption**: that every unit flow of each commodity uses 1 unit of capacity of each edge, rather than real-valued flow.

2. **No congestion assumption**: fixed capacity on each edge and cost on each edge linear in flow on that edge.

   *Counterexample that are not modeled*: commodities interact in a complicated manner such that when flow of any commodity increases on an edge, we incur increasing but non-linear cost on that edge. *e.g.* congestion on road networks is non-linear.

3. **Indivisible goods assumption**: that flow variables can be fractional, rather than integer-valued.

   **Note:** one nice feature of single-commodity network flow problems is that when supply/demand and capacity values are integer-valued, solution also integer. Not case for multi-commodity flow.

What are applications of the multi-commodity flow formulation/solution?

Last few examples of:

- data \( \rightarrow \) ?
- problem \( \rightarrow \) ?
- how technique \( \rightarrow \) ?
Examples (Straightforward):

1. Routing multiple commodities
2. Communication networks
3. Computer networks
4. Railroad transportation networks → different classes of load: passenger vs. freight
5. Distribution networks
6. Food trade networks

Warehousing Seasonal Products:

Multivessel Tanker Scheduling, e.g., crude oil, so seemingly same commodity:

- different tankers/types are different commodities
- similar to airline scheduling: vehicles in different types of airplanes in airline’s fleet

Racial Balance of Schools: network has one node for each population center, for each school, as well as a source and sink node for each ethnic group

VLSI chip design

Cutting cloth problem
Network coding is able to perform operations on flows to obtain mixing commodities and then separating them out again.

Max Flow Min Cut.
Racial Balancing

Optimal assignment of students to schools that minimizes total distance traveled by students, given specification of lower/upper bounds on required racial balance in each school.

Suppose district has $S$ schools and school $j$ has capacity $s_j$. Let $l_i$ neighborhoods, so we can compute distance from neighborhood $i$ to school $j$. Let $S_{ik}$ be # of students of color $k$ in $i$th neighborhood.

Bounds on $L_{ijk}$ and $U_{ijk}$.

\[
\begin{align*}
\sum_{i=1}^{l} L_{ijk} &= s_j, \\
\sum_{i=1}^{l} U_{ijk} &= c_k.
\end{align*}
\]

Students of $k$th color flow from source $a_k$ to sink $c_k$ via neighborhoods and schools.

Set upper bound on edge $(a_k, b_i)$ as $S_{ik}$.
Set cost of edge $(b_i, c_j)$ as $f_{ij}$ (distance between neighborhood and school).
Set capacity of edge $(c_j, r_k)$ to $s_j$.
Set upper/lower bounds on edge $(a_j, b_k)$ to $L_{ijk}$/$U_{ijk}$.