Network (graph) is collection of vertices joined by edges. \((V, E)\).

- Once in a while there can be more than one edge between two nodes, called multiedge.
- Sometimes there are self-edges.
- Simple networks have neither multiedges nor self-edges.

**Adjacency matrix** (unweighted graph)

\[
A_{ij} = \begin{cases} 
1, & \text{if edge between vertices } i \text{ and } j \\
0, & \text{otherwise.}
\end{cases}
\]

Often times edges have a strength/weight/value.

\[\rightarrow \text{ weighted graphs when } A_{ij} \text{ is strength between nodes } i \text{ and } j\]

*Weights are usually positive numbers, but they can also be negative: in social relations, positive can denote friendship whereas negative can denote animosity.*

Undirected graphs/networks have *symmetric* adjacency matrices.

Can also consider asymmetric settings with directed graphs.

Is Facebook directed or undirected? Twitter?
adjacency matrix for directed networks:

\[ A_{ij} = \begin{cases} 
1 & \text{if edge from } j \text{ to } i \\
0 & \text{otherwise.} 
\end{cases} \]

Note the convention from the second index to the first.

This can of course be weighted, directed graphs.

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Chinese Zodiac Exercise: negative weights, ok.

Special Classes of Networks

Directed Acyclic Graphs (DAG)

- citation networks, structure of proofs, conversations, ...
  (representing things that unfold in time).

- no cycles can be drawn downward according to the partial order of nodes.

Algorithm for testing whether network is acyclic:

1. find a vertex with no outgoing edges
2. if no such vertex, network is cyclic. Otherwise, if such vertex exists, remove it and all its ingoing edges from network.
3. if all vertices have been removed, network is acyclic. Otherwise, back to step 1.

Then: for every DAG, there exists at least one labeling of vertices such that adjacency matrix is strictly upper triangular.

If: suppose we construct an ordering of vertices according to the algorithm, such that all edges point downward. Then edge from \( j \) to \( i \) only if \( j > i \). That is non-zero always in upper-triangle.
Bipartite networks

a.k.a. two-mode networks.

- two kinds of vertices; edges only between vertices of different types.

Trees

- connected, undirected network with no closed loops.

- root

- leaves.

- there is exactly one path between any pair of vertices.

- n vertices, then exactly n-1 edges.

Planar Networks

- can be drawn on a plane without having any edges cross.

- consider coloring vertices of planar graph such that no two connected vertices have the same color.

- # colors required is chromatic number
four-color theorem: chromatic number of a planar graph is always four or less.

Kuratowski's Theorem:
Every non-planar network contains at least one subgraph that is an expansion of $K_5$ and $K_{3,3}$. 

![Diagram of K5 and K3,3](image-url)