Dynamical Systems

Consider a deterministic, scalar, system of a continuous-real-valued variable evolving in continuous time $t$.

A simple example of such a continuous dynamical system is described by $x(t)$ that evolves according to the differential equation:

$$\frac{dx}{dt} = f(x).$$

We also have an initial condition $x_0$, taken by $x$ at time $t_0$.

We saw last lecture of examples of linear dynamics $f(\cdot)$.

An example of a nonlinear dynamical system, which corresponds to infection processes, where $x$ is fraction of infected individuals in system is:

$$\frac{dx}{dt} = \beta x \left(1 - x\right).$$

One can also consider dynamical systems of several variables, as we will for networks; variables on network nodes.

$$\frac{dx}{dt} = f(x, y) \quad \text{and} \quad \frac{dy}{dt} = g(x, y).$$

Can also consider time-varying functions:

$$\frac{dx}{dt} = f(x, t).$$
This not needed. Formally, since we can write:

\[ \frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = 1 \]

with initial condition \( y(0) = 0 \).

So basically just introduce a new variable \( y = t \) for all time.

Another generalization is considering systems governed by higher-order derivatives

\[ \frac{d^2x}{dt^2} + ( \frac{dx}{dt} )^2 - \frac{dx}{dt} \cdot f(x). \]

Again, this can be reduced to the first-order setting by introducing new variables: \( y = \frac{dx}{dt} \), so

\[ \frac{dx}{dt} = y, \quad \frac{dy}{dt} = f(x) - y^2 + y \]

Thus we focus largely on first-order dynamics.

Simple one-variable dynamics equations can always be solved in principle:

\[ \frac{dy}{dt} = f(x) \]

as

\[ \int_{x_0}^x \frac{dx'}{f(x')} = t-t_0 \]

though integral may be difficult or unknown in closed form.

Recall “solving” means going from an implicit characterization of the system state to an explicit one.
For cases with two or more variables, it is not in general possible to find a solution. In networks, large number of variables...

One approach to analysis is through **fixed-points**.

A **fixed-point** is steady-state value of system, such that it remains stationary.

**fixed-point**: \( x = x^* \) such that \( f(x^*) = 0 \), i.e. \( \frac{dx}{dt} = 0 \).

If system is at fixed-point, says has **equilibrium**.

For two-variable systems, fixed-point is pair \((x^*, y^*)\) s.t. \( f(x^*, y^*) = 0 \) and \( g(x^*, y^*) = 0 \). So both \( \frac{dx}{dt} = \frac{dy}{dt} = 0 \).

Dynamics close to fixed-point are easier to understand than in general:

\[ x = x^* + \epsilon. \]

Then
\[ \frac{dx}{dt} = \frac{d}{dt} x^* = f(x^* + \epsilon). \]

Now perform Taylor expansion about \( x = x^* \) to \( \epsilon \):

\[ \frac{d}{dt} \epsilon = f(x^*) \cdot \epsilon f'(x^*) + O(\epsilon^2) \]

where \( f'(\cdot) \) is derivative.

Neglecting \( O(\epsilon^2) \) term and noting \( f(x^*) = 0 \),

\[ \frac{d}{dt} \epsilon = \epsilon f'(x^*). \]
This is first-order linear differential eq:

\[\varepsilon(t) = \varepsilon(0) e^{\lambda t}\]

where \[\lambda = f'(x^0)\]

Depending on sign of \(\lambda\), distance \(\varepsilon\) from fixed point will either grow or decay exponentially in time; linear stability analysis.

Attracting fixed point, \(\lambda < 0\), for which points close to fixed-point are attracted to it.

Repelling fixed point, \(\lambda > 0\).

\(\lambda = 0\), often also either attracting or repelling but need to look at higher-order terms; but strange things can also happen.

Extend linear stability analysis to multimachine systems.

\((x^0, y^0)\) when \(f(x^0, y^0) = 0\), \(g(x^0, y^0) = 0\).

Consider nearby point: \(x = x^0 + \varepsilon x\), \(y = y^0 + \varepsilon y\).

\[\frac{dx}{dt} = \frac{d\varepsilon x}{dt} = f(x^0 + \varepsilon x, y^0 + \varepsilon y)\]

\[= f(x^0, y^0) + \varepsilon x f'(x) + \varepsilon y f'(y) + \ldots\]

where \(f'(x)\) and \(f'(y)\) are the \(x\) and \(y\) derivatives.

Neglecting higher-order terms:

\[\frac{d\varepsilon x}{dt} = \varepsilon x f'(x) + \varepsilon y f'(y)\]
Similarly
\[ \frac{dx}{dt} = \varepsilon_x f^{(2)}(x^0, y^0) + \varepsilon_y f^{(3)}(x^0, y^0). \]

While in matrix-vector form:
\[ \frac{dx}{dt} = J\epsilon \quad \text{where} \quad \epsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \end{bmatrix} \]
\[ J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \]
is the Jacobian, whose all derivatives evaluated at the fixed point.

If the Jacobian is diagonal:
\[ \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial g}{\partial x} \end{bmatrix}, \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \end{bmatrix} \]
where \( \lambda_1, \lambda_2 \) are real numbers, then equations for \( \varepsilon_x \) and \( \varepsilon_y \) are decoupled.
\[ \frac{d\varepsilon_x}{dt} = \lambda_1 \varepsilon_x \quad \Rightarrow \quad \varepsilon_x(t) = \varepsilon_x(0) e^{\lambda_1 t} \Rightarrow x(t) = x^0 + \varepsilon_x(0) e^{\lambda_1 t} \]
\[ \frac{d\varepsilon_y}{dt} = \lambda_2 \varepsilon_y \quad \Rightarrow \quad \varepsilon_y(t) = \varepsilon_y(0) e^{\lambda_2 t} \Rightarrow y(t) = y^0 + \varepsilon_y(0) e^{\lambda_2 t} \]

So \( x \) and \( y \) are independently attracted/repelled, depending on sign of \( \lambda_1, \lambda_2 \).

If \( \lambda_1, \lambda_2 \) both negative \( \Rightarrow \) attractor
\( \lambda_1, \lambda_2 \) both positive \( \Rightarrow \) repeller
mixed \( \Rightarrow \) saddle point.

What if \( J \) is not diagonal?
Find combinations of variables \( x \) and \( y \) that move independently as \( x \) and \( y \) do alone.

Recall C. eigns functional subcircuits; putty on eigenglassers.

\[
\begin{align*}
Z_1 &= a_1 x + b_1 y \\
Z_2 &= a_2 x + b_2 y
\end{align*}
\]

So

\[
\begin{bmatrix}
Z_1 \\
Z_2
\end{bmatrix} =
\begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\quad \text{or} \quad Z = Q E
\]

So evolution of \( Z \) close to fixed point \( \mu \):

\[
\frac{d\mathbf{Z}}{dt} = Q \frac{dE}{dt} = QT E = QT \Lambda^{-1} E.
\]

If \( Z_1 \) and \( Z_2 \) are to evolve independently, want \( QT \Lambda^{-1} \) to be diagonal.

\( Q \) should be matrix of eigenvectors of \( T \).

\[
\frac{dZ_1}{dt} = \lambda_1 Z_1, \quad \frac{dZ_2}{dt} = \lambda_2 Z_2 \quad \text{when} \quad \lambda_1, \lambda_2 \text{ are eigenvalues of } T.
\]

So

\[
\begin{align*}
Z_1(t) &= Z_1(0) e^{\lambda_1 t} \\
Z_2(t) &= Z_2(0) e^{\lambda_2 t}
\end{align*}
\]

The eigenvector lines play the role of axes.

What if eigenvalues are complex-valued?

oscillation, spirals

Note: eigenvalues must be either real or appear in complex conjugate pair.
In addition to fixed points, also possible to have limit cycles: stable oscillatory behavior.

Also strange attractors, having fractal structure.

See also theory of chaos

dynamical systems that are highly sensitive to initial conditions.