Consider dynamics on networks as defined by differential equations.

Simplest setting is of first-order linear systems.

\[ x'_1 = p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + f_1(t) \]
\[ x'_2 = p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + f_2(t) \]
\[ \vdots \]
\[ x'_n = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + f_n(t) \]

Define a coefficient matrix
\[ P(t) = \begin{bmatrix} p_{ij}(t) \end{bmatrix} \]

and column vector \( x = [x_i]^T \) and \( f(t) = [f_i(t)]^T \).

Then the first-order linear system is:
\[ \frac{dx}{dt} = P(t)x + f(t). \]

A solution on the open interval \( I \) is a column vector function \( x(t) = [x_i(t)]^T \) such that the component functions of \( x \) satisfy the system of differential equations identically on \( I \).

If all \( p_{ij}(t) \) and \( f_i(t) \) are all continuous on \( I \), then there is a unique solution \( x(t) \) on \( I \) satisfying preassigned initial conditions \( x(a) = b \).
Consider a small example as a network:

\[ x_1' = 4x_1 - 3x_2 \]
\[ x_2' = 6x_1 - 7x_2 \]

which can be written as

\[ \frac{dx}{dt} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} x = P x. \]

Consider some conjectured solutions:

\[ x_1(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} \quad \text{and} \quad x_2(t) = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}. \]

To verify:

\[ P x_1 = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix} \] which we can check = \[ x_1' \]

\[ P x_2 = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} = \begin{bmatrix} -5e^{-5t} \\ -15e^{-5t} \end{bmatrix} \] which we can check = \[ x_2' \].

How do we generally find solutions? \( \frac{dx}{dt} = P(t) x \) is homogeneous equation, or "unforced response".

Let solution be \( x_j(t) = \begin{bmatrix} x_{j1}(t) \\ \vdots \\ x_{jn}(t) \end{bmatrix} \)

It should be linear combination of a independent solutions \( x_1, x_2, \ldots, x_n \). By principle of superposition.
Continuing example,

\[ x(t) = c_1 x_1(t) + c_2 x_2(t) = c_1 \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-3t} \\ 3e^{-3t} \end{bmatrix} \]

should also be a solution:

\[ x_1(t) = 3c_1 e^{2t} + c_2 e^{-3t} \]
\[ x_2(t) = 2c_1 e^{2t} + 3c_2 e^{-3t} \]

for general constants \( c_1, c_2 \).

We can use initial conditions to get constants in initial value problems.

Consider constant coefficient setting
and use eigenvalue method to get solutions.

\[ x_1' = a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \]
\[ x_2' = a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \]
\[ \vdots \]
\[ x_n' = a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n \]

We know solution will be look from a linearly independent solutions
\( x_1, \ldots, x_n \)

and general solution will be

\[ x(t) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \]

with arbitrary coefficients.

It seems reasonable to expect solutions of the form:

\[ x(t) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} v_1 e^{At} \\ v_2 e^{At} \\ \vdots \\ v_n e^{At} \end{bmatrix} = v e^{At} \]

where

\[ v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \]
when \( \lambda, \nu, \nu, \ldots, \nu \) are appropriate scalar constants.

This is because \( x_i = \nu_i e^{\lambda t} \) and \( x_i' = \lambda \nu_i e^{\lambda t} \)

and the \( e^{\lambda t} \) factor cancels everywhere in the differential equation.

Hence \( x_i' = \lambda x_i \) and trial solution \( x = \nu e^{\lambda t} \) with \( x_i' = \lambda \nu e^{\lambda t} \)

So
\[
\lambda \nu e^{\lambda t}, \quad \lambda \nu e^{\lambda t}
\]

\[
\lambda \nu = \lambda \nu.
\]

so the eigenvectors/eigenvalues are what is needed.

So to find solutions of \( x' =Ax \)

1. Find eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) of \( A \)

2. try to find \( n \) linearly independent eigenvectors \( v_1, v_2, \ldots, v_n \) associated with these eigenvalues.

3. If \( \# \) possible, then we get \( n \) linearly independent solutions

\[
x_1(t) = \nu_1 e^{\lambda_1 t}, \quad x_2(t) = \nu_2 e^{\lambda_2 t}, \ldots, \quad x_n(t) = \nu_n e^{\lambda_n t}.
\]

So general solution is
\[
x(t) = c_1 x_1(t) + c_2 x_2(t) + \ldots + c_n x_n(t).
\]
An important and useful class of continuous-time LTI systems are those whose input-output relations linear constant-coefficient differential equations:

\[ \sum_{k=0}^{N} a_k \frac{dy(t)}{dt} + \sum_{k=0}^{M} b_k \frac{dx(t)}{dt} = 0. \]

How do we find frequency response of such an LTI system?

**Approach 1**

Use fact complex exponentials are eigenfunctions of LTI system, specifically if \( x(t) = e^{j\omega t} \), then output is \( y(t) = H(j\omega) e^{j\omega t} \).

Substituting into algebraic manipulations then leads to solution for \( H(j\omega) \).

**Approach 2**

Use differentiation property of Fourier transform.

LTI system will satisfy

\[ y(j\omega) = H(j\omega) X(j\omega) \]

by convolution property.

\[ H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}. \]

Consider Fourier transform of differential equation:

\[ \mathcal{F} \left\{ \sum_{k=0}^{N} a_k \frac{dy(t)}{dt} \right\} = \mathcal{F} \left\{ \sum_{k=0}^{M} b_k \frac{dx(t)}{dt} \right\} \]

by linearity,

\[ \sum_{k=0}^{N} a_k \mathcal{F} \left\{ \frac{dy(t)}{dt} \right\} = \sum_{k=0}^{M} b_k \mathcal{F} \left\{ \frac{dx(t)}{dt} \right\}. \]
From differential property:

\[
\sum_{k=0}^{N} a_k (j\omega)^k y(j\omega) = \sum_{k=0}^{N} b_k (j\omega)^k x(j\omega)
\]

So

\[
y(j\omega) \left[ \sum_{k=0}^{N} a_k (j\omega)^k \right] = x(j\omega) \left[ \sum_{k=0}^{N} b_k (j\omega)^k \right]
\]

So

\[
H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{N} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}
\]

Now consider discrete-time equivalent, N-th order linear constant coefficient difference equation

\[
\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{N} b_k x[n-k]
\]

What is \( H(e^{j\omega}) \), the frequency response?

Here too, complex exponentials \( e^{j\omega} \) are eigenfunctions of LTI systems so

\[ x[n] e^{j\omega}, \quad y[n] e^{j\omega} \in H(e^{j\omega}) e^{j\omega}. \]

to get the precise result: 

\[
H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}
\]

Taking transform and applying linearity and time-shift property:

\[
\sum_{k=0}^{N} a_k e^{j\omega k} y(e^{j\omega}) = \sum_{k=0}^{N} b_k e^{j\omega k} x(e^{j\omega})
\]

So

\[
H(e^{j\omega}) = \frac{\sum_{k=0}^{N} b_k e^{j\omega k}}{\sum_{k=0}^{N} a_k e^{j\omega k}} \quad \text{polynomials in variable } e^{-j\omega}.
\]