Lecture 6: Photon Correlation Measurements

We consider how to measure the higher order field correlations. The first experiment performed outside the domain of one photon optics was the intensity correlation experiment of Hanbury-Brown and Twiss. In essence these experiments measure the joint photocount probability of detecting the arrival of a photon at time \( t \) and another photon at time \( t + \tau \). This may be written as an intensity or photon-number correlation function. Using the quantum detection theory developed by Glauber, the measured quantity is the normally ordered correlation function

\[
G^{(2)}(\tau) = \langle E^{(-)}(t) E^{(-)}(t + \tau) E^{(+)}(t + \tau) E^{(+)}(t) \rangle \\
= \langle : I(t) I(t + \tau) : \rangle \\
\propto \langle : n(t)n(t + \tau) : \rangle.
\]

where \( : \) indicates normal ordering, i.e., creation operators on the left of annihilation operators, \( I(t) \) is the intensity for analogue measurements and \( n(t) \) is the photon number in photon counting experiments.

It is useful to introduce the normalized second-order correlation function defined by (Glauber’s definition of high-order normalized correlation function)

\[
g^{(2)}(\tau) = \frac{G^{(2)}(\tau)}{|G^{(1)}(0)|^2}.
\]

For a field obeying Gaussian statistics with zero mean amplitude and no-phase dependent fluctuations,

\[
G^{(2)}(\tau) = G^{(1)}(0)^2 + |G^{(1)}(\tau)|^2,
\]

and thus

\[
g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2.
\]

Note \( G^{(1)}(\tau) \) is the Fourier transform of the spectrum of the field

\[
S(\omega) = \int_{-\infty}^{\infty} \! d\tau e^{-i\omega\tau} G^{(1)}(\tau).
\]

Hence for a field with a Lorentzian spectrum \( S(\omega) \sim 1/(\omega^2 + \gamma^2/4) \)

\[
g^{(2)}(\tau) = 1 + e^{-\gamma\tau},
\]
Figure 6.1: Hanbury-Brown and Twiss experiment and second order correlation measurement. The antibunching effect is usually used to characterize single-photon sources.

and for a field with a Gaussian spectrum \( S(\omega) \sim \exp(-\omega^2/2\gamma^2) \)

\[
g^{(2)}(\tau) = 1 + e^{-\gamma^2 \tau^2}, \tag{6.7}
\]

where \( \gamma \) is the spectral linewidth.

Now we evaluate the second-order correlation function for some quantum mechanical fields. We shall restrict our attention to a single-mode field and calculate \( g^{(2)}(0) \) and the variance in the photon number \( V(n) \)

\[
g^{(2)}(0) = \frac{\langle a^\dagger a^\dagger aa \rangle}{\langle a^\dagger a \rangle^2} = 1 + \frac{V(n) - \bar{n}}{\bar{n}^2} \tag{6.8}
\]

where \( V(n) = \langle (a^\dagger a)^2 \rangle - \langle a^\dagger a \rangle^2 \). For calculating \( g^{(2)}(\tau) \), one needs to know the system Hamiltonian.

For a coherent state, \( \rho = |\alpha\rangle \langle \alpha| \), \( g^{(2)}(0) = 1 \), and \( V(n) = \pi \) which is Poisson distribution in photon number.

For a number state, \( \rho = |n\rangle \langle n| \), \( g^{(2)}(0) = 1 - \frac{1}{n} \).
If \( g^{(2)}(\tau) < g^{(2)}(0) \) there is a tendency for photons to arrive in pairs. This situation is referred to as photon bunching. The converse situation, \( g^{(2)}(\tau) > g^{(2)}(0) \) is called antibunching. As \( g^{(2)}(\tau) \to 1 \) on a sufficiently long time scale, thus a field for which \( g^{(2)}(0) < 1 \) will always exhibit antibunching on some time scale. A value of \( g^{(2)}(0) \) less than unity could not have been predicted by a classical analysis. To obtain \( g^{(2)}(0) < 1 \) would require the field to have elements of negative probability, which is forbidden for a true probability distribution. This effect known as photon antibunching is a feature peculiar to the quantum mechanical nature of the electromagnetic field.

The even-ordered correlation functions such as the second-order correlation function \( G^{(2,2)}(x) \) contain no phase information and are a measure of the fluctuations in the photon number. The odd-ordered correlation functions \( G^{(n,m)}(x_1 \ldots x_n, x_{n+1} \ldots x_{n+m}) \) with \( n \neq m \) will contain information about the phase fluctuations of the electromagnetic field. The variances in the quadrature phases are given by measurements of this type. These schemes involve homodyning the signal field with a reference signal known as the local oscillator before photodetection. Homodyning with a reference signal of fixed phase gives the phase sensitivity necessary to yield the quadrature variances.

Consider two fields \( E_1(r, t) \) and \( E_2(r, t) \) combined on a beam splitter with
transmittivity $\eta$, as shown in Fig. 6.2a. We can use boson operators $a, b$ to characterize the two fields. The fields out of the beam splitter is

$$c = \sqrt{\eta} a + i \sqrt{1 - \eta} b.$$  \hfill (6.9)

The operator relation of a beam splitter is a direct replacement using operators in the classical amplitude relation of the beam splitter. **A few words on the beam splitter: the input of un-used port is a vacuum state, which will contribute to the quantum fluctuation at the output port!**

The mean photo-electron current in the detector is proportional to $c^\dagger c$ which is given by

$$\langle c^\dagger c \rangle = \eta \langle a^\dagger a \rangle + (1 - \eta) \langle b^\dagger b \rangle - i \sqrt{\eta(1 - \eta)} \left( \langle a \rangle \langle b^\dagger \rangle - \langle a^\dagger \rangle \langle b \rangle \right). \hfill (6.10)$$

Here $a$ and $b$ are operators in different Hilbert spaces corresponding to the two input ports (so more precisely $a \otimes b^\dagger$). When the input states are separable (tensor product states), we have $\langle ab^\dagger \rangle = \langle a \rangle \langle b^\dagger \rangle$.

Let us take the field $E_2$ to be the local oscillator and assume it to be in a coherent state of large amplitude $\beta$. Then we may neglect the first term in (6.10) and write $\langle c^\dagger c \rangle$ in the form

$$\langle c^\dagger c \rangle \approx (1 - \eta)|\beta|^2 + |\beta| \sqrt{\eta(1 - \eta)} \langle X_{\theta + \pi/2} \rangle, \hfill (6.11)$$

where

$$X_{\theta} \equiv ae^{-i\theta} + a^\dagger e^{i\theta}, \hfill (6.12)$$

and $\theta$ is the phase of $\beta$. We see that when the contribution from the reflected local oscillator intensity level is subtracted, the mean photo-current in the detector is proportional to the mean quadrature phase amplitude of the signal field defined with respect to the local oscillator phase. If we change $\theta$ through $\pi/2$ we can determine the mean amplitude of the two canonically conjugate quadrature phase operators.

The rms fluctuation current is determined by the variance of $c^\dagger c$. For an intense local oscillator in a coherent state this variance is

$$V(n_c) \approx (1 - \eta)^2 |\beta|^2 + |\beta|^2 \eta(1 - \eta) V\left( X_{\theta + \pi/2} \right). \hfill (6.13)$$

The first term here represents reflected local oscillator intensity fluctuations. If this term is subtracted out, the photo-current fluctuations are determined
by the variances in $X_{\theta+\pi/2}$, the measured quadrature phase operator. To subtract out the contribution of the reflected local oscillator field balanced homodyne detection may be used (Fig. 6.2b). In this scheme the output from both ports of the beam splitter is directed to a photodetector and the resulting currents combined with appropriate phase shifts (usually a subtraction operation) before subsequent analysis.

Mathematically, we have transmitted light in the other port

$$d = \sqrt{\eta} b + i \sqrt{1 - \eta} a. \quad (6.14)$$

If we choose $\eta = 0.5$, following similar calculation above, then the subtracted photo-current signal is proportional to

$$\langle c^\dagger c - d^\dagger d \rangle \approx |\beta| \langle X_{\theta+\pi/2} \rangle, \quad \text{and} \quad V (c^\dagger c - d^\dagger d) \approx |\beta|^2 V (X_{\theta+\pi/2}). \quad (6.15)$$

where the contribution from local oscillator is eliminated. One might note both the amplitude and variance of the signal are amplified by the local oscillator amplitude $|\beta|$, thus rendering the signal-noise ratio invariant; but this is by ignoring other noises in the system, such as electronic and thermal noises. The key advantage of balanced homodyne detection is thus to elevate the weak signal out of the background noise level without adding additional noise due to the local oscillator.