Lecture 5: Field Correlation Functions

In this lecture and next, we will describe theories related to characterization and measurements of the coherence properties of quantum optical fields. We first consider the quantum detection of an electromagnetic field, following the treatment of R. Glauber (Nobel prize 2005).

We shall assume we have an ideal detector working on an absorption mechanism which is sensitive to the field \( E^{(+)}(r, t) \) at the space-time point \((r, t)\). The transition probability of the detector for absorbing a photon at position \( r \) and time \( t \) is proportional to

\[
T_{if} = \left| \langle f | E^{(+)}(r, t) | i \rangle \right|^2
\]

if \( |i\rangle \) and \( |f\rangle \) are the initial and final states of the field. We do not, in fact, measure the final state of the field but only the total counting rate. To obtain the total count rate we must sum over all states of the field which may be reached from the initial state by an absorption process. We can extend the sum over a complete set of final states since the states which cannot be reached (e.g., states which \( |f\rangle \) differ from \( |i\rangle \) by two or more photons) will not contribute to the result since they are orthogonal to \( E^{(+)}(r, t)|i\rangle \). The total counting rate or average field intensity is

\[
I(r, t) = \sum_f T_{fi} = \sum_f \langle i | E^{(-)}(r, t) | f \rangle \langle f | E^{(+)}(r, t) | i \rangle = \langle i | E^{(-)}(r, t) E^{(+)}(r, t) | i \rangle.
\]

The above result assumes that the field is in a pure state \( |i\rangle \). The result may be easily generalized to a statistical mixture state with the probability \( P_i \), i.e.,

\[
I(r, t) = \sum_i P_i \langle i | E^{(-)}(r, t) E^{(+)}(r, t) | i \rangle = \text{Tr} \{ \rho E^{(-)}(r, t) E^{(+)}(r, t) \},
\]

where \( \rho \) is the density operator defined by \( \rho = \sum_i P_i |i\rangle \langle i| \).

These results show the counting rate is related to the field correlation function. More generally the correlation between the field at the space-time point \( x = (r, t) \) and the field at the space-time point \( x' = (r, t') \) may be written as the correlation function

\[
G^{(1)}(x, x') = \text{Tr} \{ \rho E^{(-)}(x) E^{(+)}(x') \}.
\]
The first-order correlation function of the radiation field is sufficient to account for classical interference experiments. To describe experiments involving intensity correlations such as the Hanbury-Brown and Twiss experiment, it is necessary to define higher-order correlation functions. Let’s describe the multiple photodetections. Consider a number of photo-detectors located at \( r_1, r_2, \) etc.. We ask the joint probability that detections are registered at position \( r_1 \) and at time \( t_1 \) within a short time interval \( \Delta t_1 \), at position \( r_2 \) and at time \( t_2 \) within a short time interval \( \Delta t_2 \), etc. If there are \( N \) detectors, ordered so that
\[
t_1 < t_2 < \cdots < t_N, \tag{5.5}
\]
the probability for the transition from the initial state \(|i\rangle\) to the final state \(|f\rangle\) is
\[
T_{if} = \left| \langle f \middle| E^+(r_N, t_N) \cdots E^+(r_2, t_2) E^+(r_1, t_1) \rangle i \right|^2. \tag{5.6}
\]
Following Eq. 5.2, the total counting rate
\[
I(r_1, t_1; \cdots; r_N, t_N) = \langle i \middle| E^-(r_1, t_1) \cdots E^-(r_N, t_N) E^+(r_N, t_N) \cdots E^+(r_1, t_1) \rangle i \tag{5.7}
\]
The time-ordering here is necessary, as we are working in the Heisenberg picture: operators evolve in time and operators at different time may not commute.

If we introduce the intensity operator \( \hat{I}(r, t) = E^-(r, t) E^+(r, t) \), then
\[
I(r_1, t_1; \cdots; r_N, t_N) = \langle i \middle| \mathcal{T} : \hat{I}(r_1, t_1) \cdots \hat{I}(r_1, t_1) : i \rangle, \tag{5.8}
\]
where \( : : \) means normal ordering that rearranging creation operators on the left of annihilation operators, and \( \mathcal{T} \) means time ordering that rearranging creation operators in forward time order and annihilation operators in backward time order. No commutation relations need to be applied during rearranging.

We see the total counting rate is related to higher order correlations. We define the \((n, m)\)th-order correlation function of the electromagnetic field as
\[
G^{(n,m)}(x_1 \ldots x_n, y_m \ldots y_1) = \text{Tr} \{ \rho E^-(x_1) \cdots E^-(x_n) E^+(y_m) \cdots E^+(y_1) \} \tag{5.9}
\]
If \( n = m \), we will simply use \( G^{(n)} \).
A number of interesting inequalities can be derived from the general expression

$$\text{Tr} \left\{ \rho A^\dagger A \right\} \geq 0 \quad (5.10)$$

which follows from the non-negative character of $A^\dagger A$ for any linear operator $A$. We list a few useful ones below.

$$G^{(n)} (x_1 \ldots x_n, x_n \ldots x_1) \geq 0, \quad (5.11)$$

$$\det \left[ G^{(1)} (x_i, x_j) \right] \geq 0, , \quad (5.12)$$

(For $n = 1$, $G^{(1)} (x_1, x_1) G^{(1)} (x_2, x_2) \geq \left| G^{(1)} (x_1, x_2) \right|^2$)

Cauchy–Schwartz inequality:

$$\left[ G^{(n,m)} (x_1 \ldots x_n, y_m \ldots y_1) \right]^2 \leq G^{(n)} (x_1 \ldots x_n, x_n \ldots x_1) G^{(m)} (y_1 \ldots y_m, y_m \ldots y_1) \quad (5.13)$$

For two beams, the Cauchy–Schwartz inequality is

$$\left[ G^{(2)}_{12} (t) \right]^2 \leq G^{(2)}_{11} (0) G^{(2)}_{22} (0). \quad (5.14)$$

Classical optical interference experiments correspond to a measurement of the first-order correlation function. For example, consider two fields interference

$$E^{(+)} (r, t) = E^{(+)}_1 (r, t) + E^{(+)}_2 (r, t), \quad (5.15)$$

where $E^{(+)}_i (r, t)$ is the field produced by source $i$ at position $x_i$. The intensity observed on the screen is proportional to

$$I = \text{Tr} \left\{ \rho E^{(-)} (r, t) E^{(+)} (r, t) \right\}$$

$$= G^{(1)} (x_1, x_1) + G^{(1)} (x_2, x_2) + 2 \text{Re} \left\{ G^{(1)} (x_1, x_2) \right\}, \quad (5.16)$$

where propagation phase and amplitude factors are absorbed into a normalization constant. The interference fringes arise from the oscillations of the last term.

Introducing the normalized correlation function

$$g^{(1)} (x_1, x_2) = \frac{G^{(1)} (x_1, x_2)}{\left[ G^{(1)} (x_1, x_1) G^{(1)} (x_2, x_2) \right]^{1/2}}, \quad (5.17)$$
According to Eq. 5.12, \( 0 \leq g^{(1)} (x_1, x_2) \leq 1 \). The visibility of the fringes is given by
\[
v = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}}. \tag{5.18}\]
Using 5.3 and 5.16
\[
v = \left| g^{(1)} \right| 2 \left( I_1 I_2 \right)^{1/2} \frac{1}{I_1 + I_2}. \tag{5.19}\]
\( |g^{(1)} (x_1, x_2)| \) provides a natural measure of the degree of first order coherence.

There are multiple ways to extend Eq. 5.17 to higher order correlation. One definition is as follows,
\[
g^{(n,m)} (x_1, \ldots, x_n; y_m, \ldots, y_l) = \frac{G^{(n,m)} (x_1, \ldots, x_n; y_m, \ldots, y_l)}{\prod_{r=1}^{n} [G^{(n,n)} (x_r, x_r, x_r, \ldots, x_r)]^{1/2n} \prod_{r=1}^{m} [G^{(m,m)} (y_r, y_r, y_r, \ldots, y_r)]^{1/2m}}. \tag{5.20}\]
For fields for which the phase space functional \( P(\alpha) \) in the diagonal coherent-state representation (\( P \) representation) is non-negative and has the character of a classical probability density, it may be shown that
\[
0 \leq |g^{(n,m)} (x_1, \ldots, x_n; y_m, \ldots, y_l)| \leq 1, \tag{5.21}\]
so that \( |g^{(n,m)}| \) might be interpreted as a degree of \((n,m)\)th-order coherence in certain sense. For \( n = m = 2 \) and \( x_r = y_r \), Eq. 5.20 reduces to
\[
g^{(2,2)} (x_1, x_2; x_2, x_1) = \frac{G^{(2)} (x_1, x_2; x_2, x_1)}{[G^{(2)} (x_1, x_1, x_1)]^{1/2} [G^{(2)} (x_2, x_2, x_2)]^{1/2}}. \tag{5.22}\]
Note Eq. 5.22 combined with Eq. 5.21 is different from the Cauchy–Schwartz inequality (Eq. 5.13), so it can break the upper limit of 1 for certain light with quantum characters.

There is no evidence that higher order correlations are more important than second order correlation. No examples are known of fields that are coherent to some orders higher than the second order but not others. So in practice, only second order correlation is used to characterize the quantum correlations of the fields.