Lecture 3: Squeezed States

Minimum-uncertainty states with reduced uncertainty in one quadrature at the expense of increased uncertainty in the other, e.g., $\Delta X_1 < 1 < \Delta X_2$ are called squeezed states. Mathematically, they may be generated from vacuum or coherent states by using the unitary squeeze operator

$$S(\varepsilon) = \exp \left( \frac{1}{2} \varepsilon^* a^2 - 1/2 \varepsilon a^\dagger a^2 \right),$$  \hspace{5em} (3.1)

where $\varepsilon = r e^{2i\phi}$. We will show later squeezed states can be generated in nonlinear crystals which provides a Hamiltonian equivalent to Eq. 3.1.

Note the squeeze operator obeys the relations

$$S^\dagger(\varepsilon) = S^{-1}(\varepsilon) = S(-\varepsilon),$$  \hspace{5em} (3.2)

and has the following useful transformation properties

$$S^\dagger(\varepsilon) a S(\varepsilon) = a \cosh r - a^\dagger e^{-2i\phi} \sinh r,$$

$$S^\dagger(\varepsilon) a^\dagger S(\varepsilon) = a^\dagger \cosh r - a e^{-2i\phi} \sinh r,$$

$$S^\dagger(\varepsilon) (Y_1 + iY_2) S(\varepsilon) = Y_1 e^{-r} + iY_2 e^r,$$  \hspace{5em} (3.3)

where

$$Y_1 + iY_2 = (X_1 + iX_2) e^{-i\phi}$$  \hspace{5em} (3.4)

is a rotated complex amplitude. The squeeze operator attenuates one component of the (rotated) complex amplitude, and it amplifies the other component. The degree of attenuation and amplification is determined by $r = |\varepsilon|$, which will be called the squeeze factor.

The squeezed state $|\alpha, \varepsilon\rangle$ is obtained by first squeezing the vacuum and then displacing it

$$|\alpha, \varepsilon\rangle = D(\alpha) S(\varepsilon)|0\rangle.$$  \hspace{5em} (3.5)

Equivalently,

$$|\alpha, \varepsilon\rangle = S(\varepsilon) D(\beta)|0\rangle,$$  \hspace{5em} (3.6)

where $\alpha = \beta \cosh r - \beta^* e^{2i\phi} \sinh r$.

A squeezed state has the following expectation values and variances

$$\frac{1}{2} \langle X_1 + iX_2 \rangle = \frac{1}{2} \langle Y_1 + iY_2 \rangle e^{i\phi} = \alpha,$$

$$\Delta Y_1 = e^{-r}, \quad \Delta Y_2 = e^r,$$

$$\langle N \rangle = |\alpha|^2 + \sinh^2 r,$$

$$(\Delta N)^2 = |\alpha \cosh r - \alpha^* e^{2i\phi} \sinh r|^2 + 2 \cosh^2 r \sinh^2 r.$$

(3.7)
The squeezed states are complete

\[ \int \frac{d^2 \alpha}{\pi} \left| \alpha, \varepsilon \right\rangle \left\langle \alpha, \varepsilon \right| = 1. \quad (3.8) \]

The mean amplitude of squeezed states \( \left| \alpha, \varepsilon \right\rangle \) is still the same as coherent state \( \left| \alpha \right\rangle \), but the mean photon number is increased. The squeezed state has unequal uncertainties for \( Y_1 \) and \( Y_2 \) as seen in the error ellipse. The principal axes of the ellipse lie along the \( Y_1 \) and \( Y_2 \) axes, and the principal radii are \( \Delta Y_1 \) and \( \Delta Y_2 \).

![Figure 3.1: Phase space representation of squeezed states.](image)

The squeezed states can also be visualized through the electric field associated with it. The electric field for a single mode may be written in terms of the operators \( X_1 \) and \( X_2 \) as

\[ E(r, t) = \frac{1}{\sqrt{L^3}} \left( \frac{\hbar \omega}{2 \varepsilon_0} \right)^{1/2} \left[ X_1 \sin(\omega t - \mathbf{k} \cdot \mathbf{r}) - X_2 \cos(\omega t - \mathbf{k} \cdot \mathbf{r}) \right]. \quad (3.9) \]

The variance for a minimum-uncertainty state is

\[ V(E(r, t)) = \frac{1}{L^3} \left( \frac{2 \hbar \omega}{\varepsilon_0} \right) \left[ V(X_1) \sin^2(\omega t - \mathbf{k} \cdot \mathbf{r}) + V(X_2) \cos^2(\omega t - \mathbf{k} \cdot \mathbf{r}) \right]. \quad (3.10) \]

The mean and uncertainty of the electric field at \( r = 0 \) is exhibited in Figs. 3.2–c where the line is thickened about a mean sinusoidal curve to represent the uncertainty in the electric field.
Figure 3.2: Plot of the electric field versus time showing schematically the uncertainty in phase and amplitude for (a) a coherent state, (b) a squeezed state with reduced amplitude fluctuations ($V(X_1) > V(X_2)$), and (c) a squeezed state with reduced phase fluctuations ($V(X_1) < V(X_2)$).