Lecture 2: Fock States and Coherent States

The Hamiltonian has the eigenvalues \( h\omega_k (n_k + \frac{1}{2}) \) where \( n_k \) is an integer with eigenstates written as \(|n_k\rangle\) known as number or Fock states. They are eigenstates of the number operator \( N_k = a^\dagger_k a_k \)

\[
a^\dagger_k a_k |n_k\rangle = n_k |n_k\rangle. \tag{2.1}
\]

The ground state of the oscillator (or vacuum state of the field mode) is defined by

\[
a_k |0\rangle = 0. \tag{2.2}
\]

Application of the creation and annihilation operators to the number states yield

\[
a_k |n_k\rangle = n_k^{1/2} |n_k - 1\rangle, \quad a^\dagger_k |n_k\rangle = (n_k + 1)^{1/2} |n_k + 1\rangle. \tag{2.3}
\]

The first identity can be derived from \( \langle n_k | a^\dagger_k a_k |n_k\rangle = n_k \). The state vectors for the higher excited states may be obtained from the vacuum by successive application of the creation operator

\[
|n_k\rangle = \left( \frac{a^\dagger_k}{(n_k!)^{1/2}} \right)^n_0 |0\rangle, \quad n_k = 0, 1, 2, \ldots \tag{2.4}
\]

The number states are orthogonal

\[
\langle n_k | m_k \rangle = \delta_{mn}. \tag{2.5}
\]

and complete

\[
\sum_{n_k=0}^{\infty} |n_k\rangle \langle n_k| = 1. \tag{2.6}
\]

Number states are useful representation in the case when number of photons is very small. But they are not convenient to work with when number of photons are large, such as in laser. In those cases, a more appropriate basis for many optical fields are the coherent states. We shall outline the basic properties of the coherent states below. We omit the mode index \( k \) and assume photons only occupy one spatial mode. The coherent states are most easily generated using the unitary displacement operator

\[
D(\alpha) = \exp \left( \alpha a^\dagger - \alpha^* a \right) \tag{2.7}
\]
where $\alpha$ is an arbitrary complex number. Using Baker–Campbell–Hausdorff formula, we can write $D(\alpha)$ as

$$D(\alpha) = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{-\alpha^* a}. \quad (2.8)$$

The displacement operator $D(\alpha)$ has the following properties

$$D^\dagger(\alpha) = D^{-1}(\alpha) = D(-\alpha), \quad D^\dagger(\alpha)aD(\alpha) = a + \alpha$$

$$D^\dagger(\alpha)a^\dagger D(\alpha) = a^\dagger + \alpha^* \quad (2.9)$$

which justifies why it’s called displacement operator. The coherent state $|\alpha\rangle$ is generated by operating with $D(\alpha)$ on the vacuum state

$$|\alpha\rangle = D(\alpha)|0\rangle. \quad (2.10)$$

The coherent states are eigenstates of the annihilation operator $a$. This may be proved as follows:

$$D^\dagger(\alpha)a|\alpha\rangle = D^\dagger(\alpha)aD(\alpha)|0\rangle = (a + \alpha)|0\rangle = \alpha|0\rangle. \quad (2.11)$$

Multiplying both sides by $D(\alpha)$ we arrive at the eigenvalue equation

$$a|\alpha\rangle = \alpha|\alpha\rangle. \quad (2.12)$$

Since $a$ is a non-Hermitian operator its eigenvalues $\alpha$ are complex.

The coherent state can be expanded in terms of the number states as

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum \frac{\alpha^n}{(n!)^{1/2}} |n\rangle. \quad (2.13)$$

We note that the probability distribution of photons in a coherent state is a Poisson distribution

$$P(n) = |\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!}. \quad (2.14)$$

The mean photon number of a coherent state is given by $\bar{n} = \langle \alpha | a^\dagger a | \alpha \rangle = |\alpha|^2$.

The absolute magnitude of the scalar product of two coherent states is

$$|\langle \beta |\alpha\rangle|^2 = e^{-|\alpha - \beta|^2}. \quad (2.15)$$
Thus the coherent states are not orthogonal although two states $|\alpha\rangle$ and $|\beta\rangle$ become approximately orthogonal in the limit $|\alpha - \beta| \gg 1$. The coherent states form a two dimensional continuum of states and are, in fact, overcomplete. This is intuitively obvious because the coherent states are parameterized using complex numbers but number states are parameterized with non-negative integers. One can prove completeness relation

$$\frac{1}{\pi} \int |\alpha\rangle\langle\alpha|d^2\alpha = 1.$$  \hspace{1cm} (2.16)

The coherent states have a physical significance in that the field generated by a highly stabilized laser operating well above threshold is a coherent state. They form a useful basis for expanding the optical field in problems in laser physics and nonlinear optics. The coherent states have an indefinite number of photons which allows them to have a more precisely defined phase than a number state where the phase is completely random. The product of the uncertainty in amplitude and phase for a coherent state is the minimum allowed by the uncertainty principle. In this sense they are the closest quantum mechanical states to a classical description of the field.

We may write the annihilation operator $a$ as a linear combination of two Hermitian operators

$$a = \frac{X_1 + iX_2}{2}. \hspace{1cm} (2.17)$$

$X_1$ and $X_2$, the real and imaginary parts of the complex amplitude, give dimensionless amplitudes for the modes’ two quadrature phases. One can also write

$$X_1 = a + a^\dagger, \quad X_2 = -i(a - a^\dagger), \hspace{1cm} (2.18)$$

so they obey the following commutation relation

$$[X_1, X_2] = 2i. \hspace{1cm} (2.19)$$

The corresponding uncertainty principle is

$$\Delta X_1 \Delta X_2 \geq 1, \hspace{1cm} (2.20)$$

with the variance defined as $\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$. This relation with the equals sign defines a family of minimum-uncertainty states. The coherent states are a particular minimum-uncertainty state with

$$\Delta X_1 = \Delta X_2 = 1. \hspace{1cm} (2.21)$$
The coherent state $|\alpha\rangle$ has the mean complex amplitude $\alpha$ and it is a minimum uncertainty state for $X_1$ and $X_2$, with equal uncertainties in the two quadrature phases. A coherent state may be represented by an “error circle” in a complex amplitude plane whose axes are $X_1$ and $X_2$. The center of the error circle lies at $\frac{1}{2} \langle X_1 + iX_2 \rangle = \alpha$ and the radius $\Delta X_1 = \Delta X_2 = 1$ accounts for the uncertainties in $X_1$ and $X_2$.

Figure 2.1: Phase space representation of coherent states.