Lecture 14: Weak Continuous Measurement and Back-action Noise


In discussing quantum measurements, another key notion to introduce is that of “weak measurements”, where one integrates the signal over time, gradually learning more about the system being measured. There are many good reasons why one may be interested in doing a weak measurement, rather than an instantaneous, strong, projective measurement. On a practical level, there may be limitations to the strength of the coupling between the system and the detector, which have to be compensated by integrating the signal over time. One may also deliberately opt not to disturb the system too strongly, e.g., to be able to apply quantum feedback techniques for state control. Moreover, as one reads out an oscillatory signal over time, one effectively filters away noise e.g., of a technical nature at other frequencies. Finally, consider an example like detection of the collective coordinate of motion of a micromechanical beam. Its zero-point uncertainty ground-state position fluctuation is typically on the order of the diameter of a proton. It is out of the question to reach this accuracy in an instantaneous measurement by scattering photons of such a small wavelength off the structure, since they would instead resolve the much larger position fluctuations of the individual atoms comprising the beam and induce all kinds of unwanted damage, instead of reading out the center-of-mass coordinate. The same holds true for other collective degrees of freedom. Not surprisingly, quantum noise plays a crucial role in determining the properties of a weak continuous quantum measurement. For such measurements, noise both determines the back-action effect of the measurement on the measured system and how quickly information is acquired in the measurement process.

Introduction to quantum noise

In classical physics, the study of a noisy time dependent quantity invariably involves its spectral density $S[\omega]$. The spectral density tells us the intensity of the noise at a given frequency and is directly related to the autocorrelation function of the noise. In a similar fashion, the study of quantum
noise involves quantum noise spectral densities. These are defined in a manner that mimics the classical case

\[ S_{xx}[\omega] = \int_{-\infty}^{+\infty} d\omega e^{i\omega t} \langle \hat{x}(t)\hat{x}(0) \rangle. \]  

(14.1)

Here \( \hat{x} \) is a quantum operator in the Heisenberg representation whose noise we are interested in, and the angular brackets indicate the quantum statistical average evaluated using the quantum density matrix.

As a simple introductory example illustrating important differences from the classical limit, consider the position noise of a simple harmonic oscillator having mass \( M \) and frequency \( \Omega \). The Hamiltonian is

\[ \hat{H} = \frac{\hat{p}^2}{2M} + \frac{1}{2} M\Omega^2 \hat{x}^2. \]  

(14.2)

The oscillator is maintained in equilibrium with a large heat bath at temperature \( T \) via some infinitesimal coupling, which we ignore in considering the dynamics. The solutions of the Heisenberg equations of motion are the same as for the classical case but with the initial position \( x \) and momentum \( p \) replaced by the corresponding quantum operators. It follows that the position autocorrelation function is

\[ G_{xx}(t) = \langle \hat{x}(t)\hat{x}(0) \rangle = \langle \hat{x}(0)\hat{x}(0) \rangle \cos(\Omega t) + \langle \hat{p}(0)\hat{x}(0) \rangle \frac{1}{M\Omega} \sin(\Omega t). \]  

(14.3)

Classically the second term on the right-hand side vanishes because in thermal equilibrium \( x \) and \( p \) are uncorrelated random variables. As we will see shortly for the quantum case, the symmetrized sometimes called the “classical” correlator vanishes in thermal equilibrium, just as it does classically: \( \langle \hat{x}\hat{p} + \hat{p}\hat{x} \rangle = 0 \). Note, however, that in the quantum case the canonical commutation relation between position and momentum implies there must be some correlations between the two, namely, \( \langle \hat{x}(0)\hat{p}(0) \rangle - \langle \hat{p}(0)\hat{x}(0) \rangle = i\hbar \). We find that in thermal equilibrium \( \langle \hat{p}(0)\hat{x}(0) \rangle = -i\hbar/2 \) and \( \langle \hat{x}(0)\hat{p}(0) \rangle = +i\hbar/2 \). Not only are the position and momentum correlated, but their correlator is imaginary. This occurs because the product of two non-commuting Hermitian operators is not itself a Hermitian operator. This means that, despite the fact that the position is Hermitian observable with real eigenvalues, its autocorrelation function is complex and given from Eq. (14.3) by

\[ G_{xx}(t) = x^2_{ZPF} \left\{ n_B(\hbar\Omega) e^{+i\Omega t} + [n_B(\hbar\Omega) + 1] e^{-i\Omega t} \right\}, \]  

(14.4)
where \( x_{ZPF}^2 \equiv \hbar/2M\Omega \) is the RMS zero-point uncertainty of \( x \) in the quantum ground state and \( n_B \) is the Bose- Einstein occupation factor. The complex nature of the autocorrelation function follows from the fact that the operator \( \hat{x} \) does not commute with itself at different times.

Because the correlator is complex it follows that the spectral density is not symmetric in frequency,

\[
S_{xx}[\omega] = 2\pi x_{ZPF}^2 \{ n_B(h\Omega)\delta(\omega + \Omega) + [n_B(h\Omega) + 1]\delta(\omega - \Omega) \} .
\] (14.5)

In contrast, a classical autocorrelation function is always real, and hence a classical noise spectral density is always symmetric in frequency. Note that in the high temperature limit \( k_B T \gg \hbar\Omega \) we have \( n_B(h\Omega) \sim n_B(h\Omega) + 1 \sim k_B T/\hbar\Omega \). Thus, in this limit \( S_{xx}[\omega] \) becomes symmetric in frequency as expected classically, and coincides with the classical expression for the position spectral density. The Bose-Einstein factors suggest a way to understand the frequency asymmetry of Eq. [14.5]: the positive frequency part of the spectral density has to do with stimulated emission of energy into the oscillator and the negative-frequency part of the spectral density has to do with emission of energy by the oscillator. That is, the positive-frequency part of the spectral density is a measure of the ability of the oscillator to absorb energy, while the negative-frequency part is a measure of the ability of the oscillator to emit energy.

**Imprecision noise**

We will consider the simplest example of a non-QND measurement, namely, the weak continuous measurement of the position of a harmonic oscillator. Since the position operator does not commute with the Hamiltonian, the QND criterion is not satisfied. All of our discussion of noise in the cavity system will be framed in terms of the number-phase uncertainty relation for coherent states, as we will use coherent photon states to probe the harmonic oscillator. A coherent photon state contains a Poisson distribution of the number of photons, implying that the fluctuations in photon number obey \((\Delta N)^2 = \bar{N} \)

where \( \bar{N} \) is the mean number of photons. Further, coherent states are overcomplete and states of different phase are not orthogonal to each other; this directly implies that there is an uncertainty in any measurement of the phase. For large \( \bar{N} \), this is given by

\[
(\Delta \theta)^2 = 1/4\bar{N},
\] (14.6)
which we have seen in Lecture 7 when talking about sensitivity of phase measurement using coherent states and squeezed states. Thus, large-$N$ coherent states obey the number-phase uncertainty relation $\Delta N \Delta \theta = \frac{1}{2}$. This can also be usefully formulated in terms of noise spectral densities associated with the measurement. Consider a continuous photon beam carrying an average photon flux $\overline{N}$ (time derivative of $N$). The variance in the number of photons detected grows linearly in time and can be represented as $(\Delta N)^2 = S_{NN} t$, where $S_{NN}$ is the white-noise spectral density of photon flux fluctuations. On a physical level, it describes photon shot noise, and is given by $S_{NN} = \overline{N}$. From Eq. 14.6, the imprecision in our measurement of phase due to the photon shot noise

$$S_{\theta\theta} = 1/4\overline{N}.$$  (14.7)

The above results lead us to the fundamental wave-particle relation for ideal coherent beams,

$$\sqrt{S_{NN} S_{\theta\theta}} = \frac{1}{2}.$$  (14.8)

Before we study the role that these uncertainty relations play in measurements with high-$Q$ cavities, consider the simplest case of reflection of light from a mirror without a cavity. The phase shift of the beam having wave vector $k$ when the mirror moves a distance $x$ is $2kx$. Thus, the uncertainty in the phase measurement corresponds to a position imprecision which can again be represented in terms of a noise spectral density $S_{xx}^I = S_{\theta\theta}/4k^2$. Here the superscript $I$ refers to the fact that this is noise representing imprecision in the measurement, not actual fluctuations in the position. We also need to worry about back-action: each photon hitting the mirror transfers a momentum $2k$ to the mirror, so photon shot noise corresponds to a random back-action force noise spectral density $S_{FF} = 4\hbar^2 k^2 S_{NN}$. Multiplying these together, we have the central result for the product of the back-action force noise and the imprecision,

$$S_{FF} S_{xx}^I = \hbar^2 S_{NN} S_{\theta\theta} = \hbar^2/4.$$  (14.9)

Not surprisingly, the situation considered here is as ideal as possible. We haven’t considered the effect of back-action force acting on the dynamics of the oscillator yet, which will contribute additional noise beyond the imprecision noise.
Weak continuous measurement: Measurement of oscillator position using a resonant cavity

We now turn to the story of measurement using a high-$Q$ cavity; it will be similar to the above discussion, except that we have to account for the filtering of the noise by the cavity response. The cavity is simply described as a single bosonic mode coupled weakly to electromagnetic modes outside the cavity. The Hamiltonian of the system is given by

$$\hat{H} = H_0 + \hbar \omega_c (1 + A\hat{x})\hat{a}^\dagger \hat{a} + \hat{H}_{\text{env}}$$  \hspace{1cm} (14.10)

Here $H_0$ is the unperturbed Hamiltonian of the system whose variable $\hat{x}$ (here position) is being measured, $\hat{a}$ is the annihilation operator for the cavity mode, and $\omega_c$ is the cavity resonance frequency in the absence of the coupling $A$. The term $\hat{H}_{\text{env}}$ describes the electromagnetic modes outside the cavity, and their coupling to the cavity; it is responsible for both driving and damping the cavity mode. The damping is parametrized by rate $\kappa$, which tells us how quickly energy leaks out of the cavity; we consider the case of a high-quality factor cavity, where $Q_c \equiv \omega_c/\kappa \gg 1$. The generalized back-action force conjugate to $\hat{x}$ is,

$$\hat{F}_x \equiv -\partial \hat{H}/\partial \hat{x} = -A\hbar \omega_c \delta \hat{n}.$$  \hspace{1cm} (14.11)

Turning to the interaction term in Eq. (14.10), we see that the parametric coupling strength $A$ determines the change in frequency of the cavity as the system variable $\hat{x}$ changes. We assume for simplicity that the dynamics of $\hat{x}$ is slow compared to $\kappa$. In this limit the reflected phase shift simply varies slowly in time adiabatically following the instantaneous value of $\hat{x}$. We also assume that the coupling $A$ is small enough that the phase shifts are always very small and hence the measurement is weak. Many photons will have to pass through the cavity before much information is gained about the value of the phase shift and hence the value of $\hat{x}$.

For a sufficiently weak coupling the phase shift of the reflected beam from the cavity will depend linearly on the position $x$ of the oscillator; by reading out this phase, we may thus measure $x$. The origin of back-action noise is the photon shot noise in the cavity. Now, however, this represents a random force which changes the momentum of the oscillator. During the subsequent time evolution these random force perturbations will reappear as random fluctuations in the position. Thus the measurement is not QND. This will mean that the minimum uncertainty of even an ideal measurement
is larger by exactly a factor of 2 than the “true” quantum uncertainty of the position i.e., the ground-state uncertainty. This is known as the standard quantum limit on weak continuous position detection. It is also an example of a general principle that a linear phase-preserving amplifier necessarily adds noise, and that the minimum added noise exactly doubles the output noise for the case where the input is vacuum i.e., zero-point noise. We have seen this in Lecture 8 of parametric amplification in nonlinear optics. We will give another derivation later.

We start by emphasizing that we are speaking here of a weak continuous measurement of the oscillator position. The measurement is sufficiently weak that the position undergoes many cycles of oscillation before significant information is acquired. Thus we are not talking about the instantaneous position but rather the overall amplitude and phase, or more precisely the two quadrature amplitudes describing the smooth envelope of the motion,

\[ \hat{x}(t) = \hat{X}(t) \cos(\Omega t) + \hat{Y}(t) \sin(\Omega t). \]  

One can easily show that, for an oscillator, the two quadrature amplitudes \( \hat{X} \) and \( \hat{Y} \) are canonically conjugate and hence do not commute with each other,

\[ [\hat{X}, \hat{Y}] = i\hbar/M\Omega = 2ix_{ZPF}^2. \]  

As the measurement is both weak and continuous, it will yield information on both \( \hat{X} \) and \( \hat{Y} \). As such, one is effectively trying to simultaneously measure two incompatible observables. This basic fact is intimately related to the property mentioned above, that even a completely ideal weak continuous position measurement will have a total uncertainty that is twice the zero-point uncertainty.

We are now ready to start our heuristic analysis of position detection using a cavity detector. Consider first the mechanical oscillator we wish to measure. We take it to be a simple harmonic oscillator of natural frequency \( \Omega \) and mechanical damping rate \( \gamma_0 \). For weak damping, and at zero coupling to the detector, the spectral density of the oscillator’s position fluctuations is given by Eq. 14.5 with the delta function replaced by a Lorentzian

\[ S_{xx}[\omega] = x_{ZPF}^2 \left\{ n_B(h\Omega) \frac{\gamma_0}{(\omega + \Omega)^2 + (\gamma_0/2)^2} + [n_B(h\Omega) + 1] \frac{\gamma_0}{(\omega - \Omega)^2 + (\gamma_0/2)^2} \right\} \]

When we now weakly couple the oscillator to the cavity and drive the cavity on resonance, the phase shift of the reflected beam will be proportional
Figure 14.1: Spectral density of the symmetrized output noise of a linear position detector. The oscillator’s noise appears as a Lorentzian on top of a noise floor, i.e., the measurement imprecision.

to \( x \). As such, the oscillator’s position fluctuations will cause additional fluctuations of the phase, over and above the intrinsic shot noise-induced phase fluctuations. We consider the usual case where the noise spectrometer being used to measure the noise measures the symmetric-in-frequency noise spectral density; as such, it is the symmetric-in-frequency position noise that we detect. In the classical limit \( k_B T \gg \hbar \Omega \), this is given by

\[
\bar{S}_{xx}[\omega] \equiv \frac{1}{2} (S_{xx}[\omega] + S_{xx}[-\omega]) \\
\approx \frac{k_B T}{2 M \Omega^2} \frac{\gamma_0}{(|\omega| - \Omega)^2 + (\gamma_0/2)^2} \tag{14.15}
\]

If we ignore back-action effects, we expect to see this Lorentzian profile riding on top of the background imprecision noise floor; this is shown in Fig. 14.1. If we subtract off this noise floor, the full width at half maximum of the curve will give the damping parameter \( \gamma_0 \), and the area under the experimental curve,

\[
\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{S}_{xx}[\omega] = \frac{k_B T}{M \Omega^2}, \tag{14.16}
\]
measures the temperature.

Consider now the case where the oscillator is at zero temperature. Eq. 14.14 then yields for the symmetrized noise spectral density

\[ S_{xx}^0[\omega] = x_{ZPF}^2 \frac{\gamma_0/2}{(|\omega| - \Omega)^2 + (\gamma_0/2)^2}. \] (14.17)

One might expect that one could see this Lorentzian directly in the output noise of the detector above the measurement-imprecision noise floor. However, this neglects the effects of measurement backaction. From the classical equation of motion we expect the response of the oscillator to the back-action force \( F = F_z/x_{ZPF} \) at frequency \( \omega \) to produce an additional displacement \( \delta x[\omega] = \chi_{xx}[\omega]F[\omega] \), where \( \chi_{xx}[\omega] \) is the mechanical susceptibility

\[ \chi_{xx}[\omega] \equiv \frac{1}{M} \frac{1}{\Omega^2 - \omega^2 - i\gamma_0\omega}. \] (14.18)

These extra oscillator fluctuations will show up as additional fluctuations in the output of the detector. For simplicity, we focus on this noise at the oscillator’s resonance frequency \( \Omega \). As a result of the detector’s backaction, the total measured position noise i.e., inferred spectral density at the frequency \( \Omega \) is given by

\[
S_{xx,\text{tot}}[\Omega] = S_{xx}^0[\Omega] + \frac{1}{2} \left[ S_{xx}^1[+\Omega] + S_{xx}^1[-\Omega] \right] \\
+ \frac{|\chi_{xx}[\Omega]|^2}{2} \left[ S_{FF}[+\Omega] + S_{FF}[-\Omega] \right]
\] (14.19)

The second term is the total noise added by the measurement, and includes both the measurement imprecision and the extra fluctuations caused by the back-action. Implicit in Eq. 14.19 is the assumption that the backaction noise and the imprecision noise are uncorrelated and thus add in quadrature. It is not obvious that this is correct, since in the cavity detector the back-action noise and output shot noise are both caused by the vacuum noise in the beam incident on the cavity. It turns out that there are indeed correlations, however, the symmetrized i.e., classical correlator \( \overline{S}_{\theta F} \) does vanish for our choice of a resonant cavity drive.

Assuming we have a quantum-limited detector that obeys Eq. 14.9 and
that the shot noise is symmetric in frequency, the added position noise spectral density at resonance becomes

$$S_{xx,\text{add}}[\Omega] = \left| \chi_{xx}[\Omega] \right|^2 S_{FF} + \frac{\hbar^2}{4 S_{FF}}.$$  \hspace{1cm} (14.20)

Recall that the back-action noise is proportional to the coupling of the oscillator to the detector and to the intensity of the drive on the cavity. The added position-uncertainty noise is plotted in Fig. 14.2 as a function of $S_{FF}$. We see that for high drive intensity the backaction noise dominates the position uncertainty, while for low drive intensity the output shot noise dominates.

The added noise is minimized when the drive intensity is tuned so that $S_{FF}$ is equal to $S_{FF,\text{opt}}$, with

$$S_{FF,\text{opt}} = \hbar/2 |\chi_{xx}[\Omega]| = (\hbar/2) M \Omega \gamma_0$$  \hspace{1cm} (14.21)

At the optimal coupling strength, the measurement-imprecision noise and back-action noise each make equal contributions to the added noise, yielding

$$S_{xx,\text{add}}[\Omega] = \hbar/M \Omega \gamma_0 = S_{xx}^0[\Omega].$$  \hspace{1cm} (14.22)
Thus, the spectral density of the added position noise is exactly equal to the noise power associated with the oscillator’s zero-point fluctuations. This represents a minimum value for the added noise of any linear position detector, and is referred to as the standard quantum limit on position detection. Note that this limit only involves the added noise of the detector, and thus has nothing to do with the initial temperature of the oscillator.

Figure 14.3: Spectral density of measured position fluctuations of a harmonic oscillator $\overline{S}_{xx, \text{tot}}[\omega]$ as a function of frequency $\omega$, for a detector which reaches the quantum limit at the oscillator frequency $\Omega$. We have assumed that without the coupling to the detector the oscillator would be in its ground state. The $y$ axis has been normalized by the zero-point position noise spectral density $S^0_{xx}[\omega]$, evaluated at $\omega = \Omega$.

Finally, we emphasize that the optimal value of the coupling derived above was specific to the choice of minimizing the total position noise power at the resonance frequency. If a different frequency had been chosen, the optimal coupling would have been different; one again finds that the minimum possible added noise corresponds to the ground-state noise at that frequency. It is interesting to ask what the total position noise would be as a function of
frequency, assuming that the coupling has been optimized to minimize the noise at the resonance frequency, and that the oscillator is initially in the ground state. From our results above we have

\[
S_{xx, \text{tot}}[\omega] = x_{ZPF}^2 \frac{\gamma_0/2}{(|\omega| - \Omega)^2 + (\gamma_0/2)^2} + \frac{\hbar}{2} \left( \frac{|\chi_{xx}[\omega]|^2}{|\chi_{xx}[\Omega]|} + |\chi_{xx}[\Omega]| \right) \]

\approx x_{ZPF}^2 \frac{\gamma_0}{\hbar} \left( 1 + 3 \frac{(\gamma_0/2)^2}{(|\omega| - \Omega)^2 + (\gamma_0/2)^2} \right)

(14.23)

which is plotted in Fig. 14.3. Assuming that the detector is quantum limited, one sees that the Lorentzian peak rises above the constant background by a factor of 3 when the coupling is optimized to minimize the total noise power at resonance. This represents the best one can do when continuously monitoring zero-point position fluctuations. Note that the value of this peak-to-floor ratio is a direct consequence of two simple facts which hold for an optimal coupling at the quantum limit: (i) the total added noise at resonance (back-action plus measurement imprecision) is equal to the zero-point noise and (ii) back-action and measurement imprecision make equal contributions to the total added noise.

**Standard Haus-Caves derivation of the quantum limit on a bosonic phase-preserving amplifier**

Consider the simplest case where there is only a single mode at both the input and output of the amplifier, with corresponding operators \( \hat{a} \) and \( \hat{b} \). To derive a quantum limit on the added noise of the amplifier, one uses two simple facts. First, both the input and the output operators must satisfy the usual commutation relations

\[
[\hat{a}, \hat{a}^\dagger] = 1, \quad [\hat{b}, \hat{b}^\dagger] = 1
\]

(14.24)

Second, the linearity of the amplifier and the fact that it is phase preserving (i.e., both signal quadratures are amplified the same way) implies a simple relation between the output operator \( \hat{b} \) and the input operator \( \hat{a} \)

\[
\hat{b} = \sqrt{G}\hat{a}, \quad \hat{b}^\dagger = \sqrt{G}\hat{a}^\dagger
\]

(14.25)

where \( G \) is the dimensionless photon-number gain of the amplifier. It is clear, however, that this expression cannot possibly be correct as written because
it violates the fundamental bosonic commutation relation \([\hat{b}, \hat{b}^\dagger] = 1\). We are therefore forced to write

\[
\hat{b} = \sqrt{G} \hat{a} + \hat{F}, \quad \hat{b}^\dagger = \sqrt{G} \hat{a}^\dagger + \hat{F}^\dagger
\]  

(14.26)

where \(\hat{F}\) is an operator representing additional noise added by the amplifier. It is noise associated with the additional degrees of freedom that must invariably be present in a phase-preserving amplifier.

As \(\hat{F}\) represents noise, it has a vanishing expectation value; in addition, one also assumes that this noise is uncorrelated with the input signal, implying \([\hat{F}, \hat{a}] = [\hat{F}, \hat{a}^\dagger] = 0\) and \(\langle \hat{F} \hat{a} \rangle = \langle \hat{F}^\dagger \hat{a} \rangle = 0\). Insisting that \([\hat{b}, \hat{b}^\dagger] = 1\) thus yields

\[
[\hat{F}, \hat{F}^\dagger] = 1 - G
\]  

(14.27)

From Eq. (14.26) the noise at the amplifier output \(\Delta b\) is given by

\[
(\Delta b)^2 = G(\Delta a)^2 + \frac{1}{2} \left\langle \left\{\hat{F}, \hat{F}^\dagger\right\} \right\rangle \geq G(\Delta a)^2 + \frac{1}{2} \left| \left\langle \left[\hat{F}, \hat{F}^\dagger\right] \right\rangle \right| \geq G(\Delta a)^2 + |G - 1|/2
\]  

(14.28)

The first term here is simply the amplified noise of the input, while the second term represents the noise added by the amplifier. Note that if there is no amplification, i.e., \(G = 1\), there need not be any added noise. However, in the more relevant case of large amplification \(G \gg 1\), the added noise cannot vanish. It is useful to express the noise at the output as an equivalent noise at the input by simply dividing out the photon gain \(G\). Taking the large-\(G\) limit, we have

\[
(\Delta b)^2 / G \geq (\Delta a)^2 + \frac{1}{2}
\]  

(14.29)

Thus, we have a simple demonstration that an amplifier with a large photon gain must add at least half a quantum of noise to the input signal. Equivalently, the minimum value of the added noise is equal to the zero-point noise associated with the input mode; the total output noise, referred to the input, is at least twice the zero-point input noise.

For \(G > 1\) the RHS of Eq. (14.27) is negative. Hence the simplest possible form for the added noise is

\[
\hat{F} = \sqrt{G-1} \hat{d}^\dagger, \quad \hat{F}^\dagger = \sqrt{G-1} \hat{d}
\]  

(14.30)
where \( \hat{d} \) and \( \hat{d}^\dagger \) represent a single additional mode of the system. This is the minimum number of additional degrees of freedom that must inevitably be involved in the amplification process.

Another intuitive way to understand the necessity of additional modes to have phase preserving amplifiers is by Liouville’s theorem. In classical mechanics, Liouville’s theorem requires phase-space volume to be conserved during motion. More formally, this is related to the conservation of Poisson brackets, or in quantum mechanics to the conservation of commutation relations. As a result, amplification of both quadratures of a mode cannot happen without compressing the quadratures of additional modes, which bring along noises.

**Nondegenerate parametric amplifiers**

Let’s revisit the nondegenerate parametric amplifiers using input-output formalism, for a single port cavity,

\[
\dot{\hat{a}}_S = -\left(\frac{\kappa_S}{2}\right) \hat{a}_S + \chi \hat{a}_1^\dagger - \sqrt{\kappa_S} \hat{a}_{S, \text{in}} \\
\dot{\hat{a}}_1^\dagger = -\left(\frac{\kappa_I}{2}\right) \hat{a}_1^\dagger + \chi \hat{a}_S - \sqrt{\kappa_I} \hat{a}_{1, \text{in}}
\]

In the steady state limit, and narrow bandwidth signal,

\[
\hat{a}_S = \left(2\lambda/\kappa_S\right) \hat{a}_1^\dagger - \left(2/\sqrt{\kappa_S}\right) \hat{a}_{S, \text{in}} \\
\hat{a}_1^\dagger = \left(2\lambda/\kappa_I\right) \hat{a}_S - \left(2/\sqrt{\kappa_I}\right) \hat{a}_{1, \text{in}}
\]

And standard relation

\[
\hat{a}_{S, \text{out}} = \hat{a}_{S, \text{in}} + \sqrt{\kappa_S} \hat{a}_S
\]

We have

\[
\hat{a}_{S, \text{out}} = \frac{Q^2 + 1}{Q^2 - 1} \hat{a}_{S, \text{in}} + \frac{2Q}{Q^2 - 1} \hat{a}_{1, \text{in}}
\]

where \( Q \equiv 2\lambda/\sqrt{\kappa_I\kappa_S} \) is proportional to the pump amplitude and inversely proportional to the cavity decay rates. We have to require \( Q^2 < 1 \) to make sure that the parametric amplifier does not settle into self-sustained oscillations, i.e., it works below threshold. Under that condition, we can define the photon-number gain \( G_0 \) via

\[
-\sqrt{G_0} = \left(\frac{Q^2 + 1}{Q^2 - 1}\right)
\]

such that

\[
\hat{a}_{S, \text{out}} = -\sqrt{G_0} \hat{a}_{S, \text{in}} - \sqrt{G_0 - 1} \hat{a}_{1, \text{in}}
\]
As a result, the input-output relation Eq. \[14.36\] is precisely of the Haus-Caves form \[14.26\] and \[14.30\] for an ideal quantum-limited amplifier.

**Back-action evasion and noise-free amplification**

To amplify a single quadrature of some time dependent signal, there need not be any added noise from the measurement. Unlike the case of amplifying both quadratures, Liouville’s theorem does not require the existence of any additional degrees of freedom when amplifying a single quadrature: phase space volume can be conserved during amplification simply by contracting the unmeasured quadrature. As no extra degrees of freedom are needed, there need not be any extra noise associated with the amplification process.

Perhaps the simplest example of a phase nonpreserving or phase sensitive amplifier is the degenerate parametric amplifier. As we have shown in Lecture 7, the resulting dynamics causes one signal quadrature to be amplified while the other is attenuated, in such a way that it is not necessary to add extra noise to preserve the canonical commutation relations. Using input-output formalism,

\[
\dot{\hat{a}}_S = -\left(\frac{\kappa_S}{2}\right)\hat{a}_S + \chi\hat{a}_S^\dagger - \sqrt{\kappa_S}\hat{a}_{S,\text{in}}. \tag{14.37}
\]

For the quadrature \(\hat{x}_S = \hat{a}_S^\dagger + \hat{a}_S, \quad \hat{y}_S = i\left(\hat{a}_S^\dagger - \hat{a}_S\right)\), the output and input relation is thus

\[
\hat{x}_{S,\text{out}} = \sqrt{G}\hat{x}_{S,\text{in}}, \quad \hat{y}_{S,\text{out}} = \hat{y}_{S,\text{in}} / \sqrt{G}, \tag{14.38}
\]

where

\[
G = \left[\frac{(\chi + \kappa_S/2)}{(\chi - \kappa_S/2)}\right]^2. \tag{14.39}
\]