Homework 4  
(Due Mar. 26th)

4.1 Consider the master equation for a damped harmonic oscillator (i.e., a single mode of the electromagnetic field) as given by Eq. 10.7
(a) Show that the master equation preserves the trace of $\hat{\rho}$
(b) In steady state, show that the density operator $\hat{\rho}_{th}$ for a field in thermal equilibrium satisfies the master equation as above. Here, $\hat{\rho}_{th}$ is given by

\[
\exp\left[\frac{-\hat{H}_0}{k_B T}\right] \frac{\text{Tr}\left[\exp\left[\frac{-\hat{H}_0}{k_B T}\right]\right]}{1 - \exp\left[\frac{-\hbar \omega}{k_B T}\right]}.
\]

(1)

c) From the master equation, derive and solve the equations of motion for the mean mode amplitude $\langle \hat{a}(t) \rangle$ and for the mean photon number $\langle \hat{n}(t) \rangle$. Assume some arbitrary initial values $\langle \hat{a}(0) \rangle$ and $\langle \hat{n}(0) \rangle$, and discuss how the field comes into thermal equilibrium.

4.2 The interaction picture master equation for a damped harmonic oscillator driven by a resonant linear force is

\[
\frac{d\rho}{dt} = i\varepsilon \left[a + a^{\dagger}, \rho\right] + \frac{\gamma}{2} \left(2a\rho a^{\dagger} - a^{\dagger}a\rho - \rho a^{\dagger}a\right)
\]

(2)

Show that the steady state solution is the coherent state $|2i\varepsilon/\gamma\rangle$.

4.3 In the Jaynes–Cummings model, show that if the atom begins in the ground state and the field begins in the state $|\phi\rangle = \sum f_n |n\rangle$, the state at time $t > 0$ is the entangled state

\[
|\psi(t)\rangle = |\phi_g(t)\rangle|g\rangle + |\phi_e(t)\rangle|e\rangle
\]

(3)

where

\[
|\phi_g(t)\rangle = \sum_n f_n \cos (\Omega_{n-1} t) |n\rangle,
\]

\[
|\phi_e(t)\rangle = i \sum_n f_n \sin (\Omega_{n-1} t) |n\rangle,
\]

(4)

where $\Omega_n = g\sqrt{n+1}.$