Problem 1

Consider the transfer function

\[ H(s) = \frac{1}{s + a}, \]

where \( a > 0 \). Prove that the Nyquist plot of \( H \) is a circle of radius \( \frac{1}{2a} \) centered at the point \( \left( \frac{1}{2a}, 0 \right) \).

For any \( \omega \), we have

\[
H(j\omega) = \frac{1}{j\omega + a} = \frac{\omega}{(j\omega + a)(-j\omega + a)} = \frac{\omega}{a^2 + \omega^2}.
\]

Therefore, the Nyquist plot of \( H \) has the parametric form

\[
\left( \Re H(j\omega), \Im H(j\omega) \right) = \left( \frac{a}{a^2 + \omega^2}, -\frac{\omega}{a^2 + \omega^2} \right), \quad -\infty < \omega < \infty.
\]

Recall that the points \((x, y)\) that lie on a circle of radius \( r \) centered at the point \((a, b)\) satisfy

\[(x - a)^2 + (y - b)^2 = r^2.\]
Thus, we compute
\[
\left( \text{Re} H(j\omega) - \frac{1}{2a} \right)^2 + \left( \text{Im} H(j\omega) \right)^2
= \left( \frac{a}{a^2 + \omega^2} - \frac{a^2 + \omega^2}{2a(a^2 + \omega^2)} \right)^2 + \left( \frac{\omega}{a^2 + \omega^2} \right)^2
= \left( \frac{a^2 - \omega^2}{2a(a^2 + \omega^2)} \right)^2 + \left( \frac{2a\omega}{2a(a^2 + \omega^2)} \right)^2
= \left( \frac{1}{2a} \right)^2 \left[ (a^2 - \omega^2)^2 + 4a^2\omega^2 \right]
= \left( \frac{1}{2a} \right)^2 \frac{a^4 - 2a^2\omega^2 + \omega^4 + 4a^2\omega^2}{(a^2 + \omega^2)^2}
= \left( \frac{1}{2a} \right)^2 \frac{a^4 + 2a^2\omega^2 + \omega^4}{(a^2 + \omega^2)^2}
= \left( \frac{1}{2a} \right)^2 .
\]
This proves the claim.

**Problem 2**

For the two plant transfer functions given below, use the Nyquist stability criterion to determine all values of the feedback gain $K$ that stabilize the closed-loop system.

(a) 
\[
G(s) = \frac{1}{(s + 2)(s + 5)}
\]
\[ G_m = \text{Inf dB (at Inf rad/s)}, \quad P_m = \text{Inf} \]

\[ \omega \to 0 \Rightarrow |G(j\omega)| = \frac{1}{10}, \angle G(j\omega) = 0^\circ \]

\[ \omega = \sqrt{10} \Rightarrow |G(j\omega)| = \frac{1}{7\sqrt{10}}, \angle G(j\omega) = -90^\circ \]

\[ \omega \to \infty \Rightarrow |G(j\omega)| = 0, \angle G(j\omega) = -180^\circ \]
P: #RHP open loop poles = 0
Z: # of closed loop RHP poles
N: # of encirclements of $-1/K$

$N = Z - P \Rightarrow N = Z$

If $K > 0 \Rightarrow \frac{-1}{K} < 0$, then $N = 0$ according to the Nyquist plot
If $-10 < K \leq 0 \Rightarrow \frac{-1}{K} > \frac{1}{10}$, then $N = 0$
If $K < -10 \leq 0 \Rightarrow 0 < \frac{-1}{K} < \frac{1}{10}$, $N = 1 \Rightarrow Z = 1 \Rightarrow$
unstable closed-loop
Therefore, we need $K \geq -10$ for stability, which agrees with the Routh’s criterion.

(b) 

$$G(s) = \frac{1}{(s + 2)(s^2 + 2s + 5)}$$
From the Bode plot

\[
\omega \to 0 \Rightarrow |G(j\omega)| = \frac{1}{10}, \angle G(j\omega) = 0^\circ
\]

\[
\omega = \sqrt{\frac{5}{2}} \Rightarrow |G(j\omega)| = 0.097, \angle G(j\omega) = -90^\circ
\]

\[
\omega = 3 \Rightarrow |G(j\omega)| = 0.0385, \angle G(j\omega) = -180^\circ
\]

\[
\omega \to \infty \Rightarrow |G(j\omega)| = 0, \angle G(j\omega) = -270^\circ
\]
\[ P = 0 \]
\[ \therefore Z = N \]
Following a similar procedure as we did for part a, we will find that \(-10 < K < 26\) yields \(N = 0 \Rightarrow \) Stability

Check with Routh’s Criterion
Characteristic Equation: \(1 + KG(S) = 0 \Rightarrow s^3 + 4s^2 + 9s + 10 + K\)
Necessary condition: \(K > -10\)

\[
\begin{array}{ccc}
  s^3 & 1 & 9 \\
  s^2 & 4 & 10 + K \\
  s^1 & -\frac{1}{4}(10 + K - 36) \\
  s^0 & 10 + K \\
\end{array}
\]

Therefore, \(-\frac{1}{4}(10 + K - 36) > 0 \Rightarrow K < 26\)

**Problem 3**

For the two transfer functions and gain values given below, use the Nyquist plot to find the gain and the phase margins:
(a) \( G(s) = \frac{1}{(s - 1)(s + 2)(s + 4)}, \quad K = 10 \)

\[
KG_1(s) = \frac{10}{(s - 1)(s + 2)(s + 4)} KG_1(j\omega) = \frac{10}{(j\omega - 1)(j\omega + 2)(j\omega + 4)}
\]

Bode plot of \( KG_1(s) \)

From the bode plot, we can see that
\[ \omega \to 0 \Rightarrow |G(j\omega)| = \frac{10}{8}, \quad \angle G(j\omega) = -180^\circ \]

Therefore, \( M_{180^\circ} = \frac{10}{8} \Rightarrow GM = \frac{1}{M_{180^\circ}} = \frac{8}{10} = 0.8 = -1.94 \text{ dB}, \) which agrees with the gain margin obtained from MATLAB.

To obtain PM from the Nyquist plot, draw a unit circle and marked the points where the unit circle intersects the Nyquist plot. Draw a line from the origin to one of the points. The angle formed between that line and the \(-180^\circ\) axis is the PM. See the plot below for an illustration of the method.

Nyquist plot of \( KG_1(s) \)
PM can be computed from the \( \omega \) that makes \( |KG_1(j\omega)| = 1 \)

\[
\begin{align*}
\left| \frac{10}{(j\omega - 1)(j\omega + 2)(j\omega + 4)} \right| &= 1 \\
\left| \frac{10}{(-5\omega^2 - 8) + (-\omega^3 + 2\omega)j} \right| &= 1 \\
\left| \frac{10}{\sqrt{(-5\omega^2 - 8)^2 + (-\omega^3 + 2\omega)^2}} \right| &= 1 \\
(-5\omega^2 - 8)^2 + (-\omega^3 + 2\omega)^2 &= 100
\end{align*}
\]

Solve this equation and find that \( \omega = 0.625 \) satisfies this equation.
Substitute, \( \omega = 0.625 \) in the \( \angle KG_1(j\omega) \), which will result in \( \angle KG_1(j0.625) = -174.23^\circ \).

\[ \therefore PM = -174.23^\circ - (-180^\circ) = 5.77^\circ \], which agrees with the gain margin obtained from MATLAB.

(b) \( G(s) = \frac{1}{(s + 1)^3}, \quad K = 3 \)
\[
KG_2(s) = \frac{3}{(s + 1)^3}
\]

\[
KG_2(j\omega) = \frac{3}{(j\omega + 1)^3} = \frac{3}{(1 - 3\omega^2) + (3\omega - \omega^3)j}
\]

Bode plot of \( KG_2(s) \)

\[
\omega \to 0 \Rightarrow |G(j\omega)| = 3, \angle G(j\omega) = 0^\circ
\]

\[
\omega = \frac{1}{\sqrt{3}} \Rightarrow |G(j\omega)| = 1.95, \angle G(j\omega) = -90^\circ
\]

\[
\omega = \sqrt{3} \Rightarrow |G(j\omega)| = 0.375, \angle G(j\omega) = -180^\circ
\]

\[
\omega \to \infty \Rightarrow |G(j\omega)| = 0, \angle G(j\omega) = -270^\circ
\]

Therefore, \( M_{180^\circ} = 0.375 \Rightarrow GM = \frac{1}{M_{180^\circ}} = \frac{1}{0.375} = 2.667 \) dB, which agrees with the gain margin obtained from MATLAB.
Follow the same step as the previous problem, to obtain PM Nyquist plot of $KG_2(s)$

Solve this equation and find that $\omega = 1.04$ satisfies this equation.
Substitute, $\omega = 1.04$ in the $\angle KG_2(j\omega)$, which will result in $\angle KG_2(j1.04) = -138.36^\circ$.

$\therefore PM = -138.36^\circ - (-180^\circ) = 41.7^\circ$, which agrees with the gain margin obtained from MATLAB.