Problem 1. Calculate the magnitude and phase of the following complex numbers.

(a) $x = 4 - 3j$

(b) $x = 21 - 20j$

(c) $x = a^2/b$, where the magnitudes $|a|$ and $|b|$ are given, as well as the phases $\angle a$ and $\angle b$.

Solution.

(a) For $x = 4 - 3j$ we have that $|x| = 5$ and $\theta = \angle x$ satisfies $\cos \theta = \frac{4}{5}$ and $\sin \theta = -\frac{3}{5}$. This means $\theta = -36.9$ degrees/$-0.643$ radians.

(b) For $x = 21 - 20j$, we have that $|x| = \sqrt{431} = 29$ and $\theta = \angle x$ satisfies $\cos \theta = \frac{21}{29}$ and $\sin \theta = -\frac{20}{29}$. This means $\theta = -43.6$ degrees/$-0.761$ radians.

(c) For $x = a^2/b$, where the magnitudes $|a|$ and $|b|$ are given, as well as the phases $\angle a$ and $\angle b$ we have that $|x| = \frac{|a|^2}{|b|} = |a|^2/|b|$. If $\phi_a = \angle a$ and $a^2 = |a|^2 e^{i\phi_a}$ then $x = \frac{|a|^2}{|b|} e^{i(2\phi_a - \phi_b)}$ and so $\angle x = 2\angle a - \angle b$.

Note: The phase of a complex number $x + yj$ is well defined up to multiples of 360 degrees/2$\pi$ radians as the angle made by the vector $(x, y)$ with the positive real axis in the plane, with the counterclockwise direction considered positive. However the tan function has a period of 180 degrees/$\pi$ radians and so just saying the phase of $4 - 3j$ is $-\arctan 0.75$ does not determine the phase completely; in fact $-4 + 3j$ would have the same phase as $4 - 3j$ if that were the case, but the phase difference between these two complex numbers are 180 degrees/$\pi$ radians. One needs to check which quadrant the complex number lies in explicitly to determine the phase.

Problem 2. Calculate the following Laplace transforms $F_i = \mathcal{L}\{f_i\}$ by hand:

(a) $f_1(t) = 2 \cos(t) + \sin(t)$

(b) $f_2(t) = e^{-3t}$

(c) $f_3(t) = 2 \cos(t) + \sin(t) + e^{-3t}$

Hint: Recall Euler’s formula.

Calculate $\lim_{t \to \infty} f_i(t)$, if it exists, for the 3 previous functions. For which functions can the consequence of the Final Value Theorem be used?

Solution.

Recall that $\mathcal{L}\{e^{at}\} = \frac{1}{s - a}$ for all $a \in \mathbb{C}$, and $e^{i\theta} = \cos \theta + i \sin \theta$ (Euler’s formula). Therefore,

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

(a) Now,

$$2 \cos(t) + \sin(t) = (1 + \frac{1}{2i})e^{it} + (1 - \frac{1}{2i})e^{-it}$$

$$\implies \mathcal{L}\{f_1(t)\} = (1 + \frac{1}{2i}) \frac{1}{s - i} + (1 - \frac{1}{2i}) \frac{1}{s + i} = \frac{2s + 1}{s^2 + 1}.$$

and $\lim_{t \to \infty} f_1(t)$ does not exist, since $f_1(n\pi) = 2(-1)^n$ for all $n \in \mathbb{N}$. FVT does not apply here because $sF_1(s)$ has poles $\pm i$ and hence is not stable.
(b) Here

\[ L\{f_2(t)\} = \frac{1}{s+3} , \]

and \( \lim_{t \to \infty} f_2(t) = 0 \). FVT does apply here because \( sF_2(s) \) has only one pole at \(-3\) and hence is stable. Clearly, \( \lim_{s \to 0} sF_2(s) = 0 \) as well.

(c) Lastly,

\[ L\{f_3(t)\} = L\{f_1(t) + f_2(t)\} \]

\[ = \frac{2s + 1}{s^2 + 1} + \frac{1}{s + 3} = \frac{3s^2 + 7s + 4}{s^3 + 3s^2 + s + 3} , \]

and \( \lim_{t \to \infty} f_3(t) \) does not exist, since if it did have a limit then \( f_1(t) \) would also have a limit as \( t \to \infty \) since \( \lim_{t \to \infty} f_2(t) \) exists. FVT does not apply here because \( sF_3(s) \) has poles \( \pm i, -3 \) and hence is not stable.

**Problem 3.** Compute the step responses of the following transfer functions by hand:

(a) \( H_1(s) = \frac{4}{s + 10} \)

(b) \( H_2(s) = \frac{4}{s - 10} \)

Recall that a step response is the output of the system when all the initial conditions are zero and input is as follows:

\[ u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \]

Calculate the steady-state value of each step response.

**Solution.** (a) The Laplace transform of the step response is \( \frac{4}{s(s + 10)} \), which can be rewritten as

\[ \frac{4}{s(s + 10)} = \frac{4}{10} \left( \frac{1}{s} - \frac{1}{s + 10} \right) . \]

So the step response

\[ y(t) = \frac{4}{10} \left( \mathcal{L}^{-1} \left( \frac{1}{s} \right) - \mathcal{L}^{-1} \left( \frac{1}{s + 10} \right) \right) = \frac{4}{10} \left( 1 - e^{-10t} \right) , \]

and its steady state value is \( \frac{4}{10} \).

(b) The Laplace transform of the step response is \( \frac{4}{s(s + 10)} \), which can be rewritten as

\[ \frac{4}{s(s - 10)} = \frac{4}{10} \left( \frac{1}{s} - \frac{1}{s - 10} \right) . \]

So the step response

\[ y(t) = \frac{-4}{10} \left( \mathcal{L}^{-1} \left( \frac{1}{s} \right) - \mathcal{L}^{-1} \left( \frac{1}{s - 10} \right) \right) = \frac{-4}{10} \left( 1 - e^{10t} \right) , \]

and its steady state value does not exist.
Problem 4. Consider the following transfer functions:

(a) \( H_1(s) = \frac{1}{s^2 - s + 3} \)

(b) \( H_2(s) = \frac{s - 3}{s^2 + 5s + 4} \)

(c) \( H_3(s) = \frac{3s + 2}{s^2 + 8s + 7} \)

(d) \( H_4(s) = \frac{4.5}{s^2 + 6s + 25} \)

Calculate \( \lim_{s \to 0} H_i(s) \). Use the MATLAB command `step` to plot their step responses, and attach the plots. (The command `ltiview` may also be useful.) Can the Final Value Theorem be invoked? What is the DC gain?

Solution.

(a) Here \( \lim_{s \to 0} H_1(s) = 1/3 \). See Figure 1(a) for the step response. Since \( H_1(s) \) has poles \( (1/2 \pm \sqrt{11}/2i) \) in the RHP, the FVT does not apply and the DC gain is not defined as seen from the plot.
(b) Now \( \lim_{s \to 0} H_2(s) = -3/4 \). See Figure 1(b) for the step response. The FVT does apply here since both poles \((-4, -1)\) lie in the LHP. The DC gain is \(-0.75\), which can be seen both in the plot and by FVT.

(c) The \( \lim_{s \to 0} H_3(s) = 2/7 \). See Figure 1(c) for the step response. The FVT does apply here since both poles \((-7, -1)\) lie in the LHP. The DC gain is \(0\), which can be seen both in the plot and by FVT.

(d) Finally \( \lim_{s \to 0} H_4(s) = 9/50 \). See Figure 1(d) for the step response. The FVT does apply here since both poles \((-3 \pm \sqrt{4}i)\) lie in the LHP. The DC gain is \(0\), which can be seen both in the plot and by FVT.

**Note:** The FVT says that if a signal has transfer function \( Y(s) \) and \( sY(s) \) has all poles in the LHP, then \( \lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) \). The poles being in the left half plane (LHP) mean that all the poles have negative real parts, not that the poles themselves are negative reals, or just reals.

**Note:** To find the DC gain of a system with transfer function \( H(s) \), we apply the FVT to the Laplace transform of its step response \( H(s) \), and so \( H(s) \) has to have all poles in the LHP, not just \( sH(s) \).

**Note:** The DC gain of a system is the steady state value of its step response, not \( \lim_{s \to 0} H(s) \).

If \( H(s) \) is stable, then by the FVT the two values coincide.

**Problem 5.** Consider a linear system with transfer function \( G(s) = \frac{4}{s + 2} \). Let \( u \) denote the input and \( y \) denote the output. The response of \( G(s) \) with \( u(t) = 2 \) for \( t \geq 0 \), \( u(t) = 0 \) for \( t < 0 \), and zero initial conditions is \( y(t) = 4(1 - e^{-2t}) \).

(a) What is the response \( y_A \) from zero initial conditions if \( u_A(t) = -1 \) for \( t \geq 0 \)?

(b) What is the response \( y_A \) from zero initial conditions if \( u_A(t) = e^{-3t} \) for \( t \geq 0 \)? (Hint: As emphasized in the class, the derivations for this example in the slide for Lecture 2 are not correct. Apply the method of partial fraction to get the correct result.)

**Solution.**

(a) The response is \( y_A(t) = -2 + 2e^{-2t} \) (steps skipped, see next problem).

(b) The Laplace transform of the input now is \( U_A(s) = \frac{1}{s + 3} \) implying that the Laplace transform of the output is

\[
Y(s) = G(s)U(s) = \frac{4}{(s + 2)(s + 3)} = \frac{4}{s + 2} - \frac{4}{s + 3}
\]

which can be inverse Laplace transformed to yield \( y_A(t) = 4e^{-2t}(-1 + e^t) \).

**Problem 6.** Recall the dynamics for the mass-spring system from lecture, as depicted in Figure 2. The dynamics are:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-\frac{k}{m} & -\frac{\rho}{m}
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} u
\]

\[
y = \begin{bmatrix}
1 & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

Here, \( k \) is the spring constant and \( \rho \) is the friction coefficient.
Figure 2: The mass-spring system discussed in lecture.

(a) Find the transfer function of this system.

(b) Suppose that the $C$ matrix is replaced, such that:

$$y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Recalculate the transfer function with this sensor model. Which values of $c_1$ and $c_2$ guarantee the new system has the form of the prototypical 2nd-order response discussed in class? Write $\omega_n$ and $\zeta$ in terms of $k, \rho, m$.

Determine the steady-state response of this new system to the external force $u(t) = \cos(\omega t)$, where $\omega$ is a chosen constant.

Sketch or print plots of the magnitude and phase shift of the steady state response as functions of the input frequency $\omega$. What happens when the input frequency equals the system’s natural frequency, i.e. $\omega = \omega_n$? (This phenomena is known as resonance.)

Solution.

(a) The transfer function is readily given by the formula $H(s) = C(sI - A)^{-1}B$ to be

$$H(s) = \frac{1}{k + s(ms + \rho)}$$

(b) The new transfer function is given by:

$$H(s) = \frac{c_2 s + c_1}{k + ms^2 + \rho s}$$

and it is clear that a choice of $c_2 = 0$ and $c_1 = k$ will put the system in the standard second order format with

$$\omega_n^2 = \frac{k}{m} \quad \text{and} \quad \zeta = \frac{\rho}{2m} \sqrt{\frac{m}{k}}$$

We have that $U(s) = \frac{s}{s^2 + \omega^2}$ yielding the Laplace transform of the output $y(t)$ as

$$Y(s) = \frac{ks}{(s^2 + \omega^2)(k + s(ms + \rho))}$$

Define as

$$\Gamma := k^2 - 2km\omega^2 + m^2\omega^4 + \rho^2\omega^2$$
Then the PFE of the quantity above is given by:

\[
\frac{Y(s)}{k} = \frac{s(m^2\omega^2 - km) - kp}{\Gamma(k + ms^2 + ps)} + \frac{s(k - m\omega^2) + \rho\omega^2}{\Gamma(s^2 + \omega^2)}
\]

Considering the form of the denominators, it is clear that the steady state response arises from the latter term whereas the first term contributes a transient (why?). Computing the inverse Laplace transform of the second term one gets,

\[
y_{ss}(t) = \frac{k^2\cos(t\omega) - km\omega^2\cos(t\omega) + \rho k\omega \sin(t\omega)}{\Gamma}
\]

To elicit the relationship between \(\omega\) and the amplitude and phase of the steady state response, follow the lecture notes and handbook to get,

\[
y(t) = ||G(j\omega)|| \cos(\omega t + \angle G(j\omega))
\]

where

\[
||G(j\omega)|| = \frac{k}{\sqrt{(k - m\omega^2)^2 + \rho^2\omega^2}} \quad \text{and} \quad \angle G(j\omega) = \arctan\left(\frac{\rho\omega}{m\omega^2 - k}\right)
\]

The required sketches are as in Figure 3. Since we set \(\omega_n = 1\) we can see that as \(\omega \to \omega_n\) there is an increase in the amplitude of the response. When \(\omega = \omega_n\) this increase reaches a maximum in a phenomenon termed resonance. Resonance can be both a desirable or undesirable property in the system. See: this link.

**Problem 7.** Sketch the response \(y(t)\) vs. \(t\) for the system, initial conditions, and input given below. Label the steady-state value of \(y\) and the approximate settling time. Also label the approximate peak value of \(y\). Specify whether the system is over or underdamped.

\[
\ddot{y} + 2\dot{y} + 25y = 12.5u, \quad (1)
\]

\[
y(0) = 0, \quad \dot{y}(0) = 0, \quad u(t) = \begin{cases} 0 & t < 0 \text{ sec} \\ 4 & t \geq 0 \text{ sec} \end{cases} \quad (2)
\]

**Solution.**

A straightforward Laplace transform of the given differential equation with initial conditions and prescribed input yields,

\[
Y(s) = \frac{2}{s} - \frac{2(s + 2)}{s^2 + 2s + 25}
\]
which can be inverse transformed to give
\[ y(t) = \frac{1}{6} e^{-t} \left( 12e^t - \sqrt{6} \sin \left( 2\sqrt{6}t \right) - 12 \cos \left( 2\sqrt{6}t \right) \right) \]

and a plot is shown in Figure 4 with settling and peak times marked on the x-axis and approximate peak value and steady state value marked on the y-axis. The system is an underdamped one. Note that the plot above can be sketched without solving the inverse transform. To wit, the steady state value can be found by an application of the FVT to the Laplace Transform above. Further, writing \( Y(s) = \frac{s}{2} \cdot H(s) \) where,

\[ H(s) := \frac{25}{s^2 + 2s^2 + 25} \]

allows representing the given system as a standard second order system response to twice the unit step function. This gives,
\( \omega_n = 5, \quad \zeta = 1/5, \quad \omega_d = \omega_n \sqrt{1 - \zeta^2} = 2\sqrt{6} \)
and \( \sigma = \zeta \omega_n = 1 \). Therefore we can calculate the quantities marked on the plot above,
\[ t_p = \frac{\pi}{\omega_d} = \frac{\pi}{2\sqrt{6}}, \quad t_s \approx \frac{3}{\sigma} = 3 \text{ sec}, \quad y_p(t) = (1 + M_p) \cdot y_{ss}(t) \]

where
\[ M_p = \exp \left( -\frac{\pi \zeta}{\sqrt{1 - \zeta^2}} \right) = e^{-\frac{\pi}{2\sqrt{6}}} \]

**Problem 8.** Consider the transfer function:
\[ H(S) = \frac{25}{S^2 + 6S + 25} \]

(a) Suppose you are given the following time-domain specs: rise time \( t_r \leq 0.6 \) and settling time \( t_s \leq 1.6 \). (Here we’re considering settling time to within 5% of the steady-state value.) Plot the admissible pole locations in the s-plane corresponding to these two specs. Does this system satisfy these specs?

(b) Repeat the previous problem for the specs: rise time \( t_r \leq 0.6 \), settling time \( t_s \leq 1.6 \), and magnitude \( M_p \leq 1/e^2 \). Plot the admissible pole locations; does this system satisfy these specs?
(c) Repeat the previous problem for the specs: rise time \( t_r \leq 0.6 \), settling time \( t_s \leq 1.6 \), and peak time \( t_p \leq 1 \). Plot the admissible pole locations; does this system satisfy these specs?

Solution.

![Figure 5: From left to right, A, B and C - solutions to Problem 8](image)

(a) Given that \( t_r = \frac{1.8}{\omega_n} \) and \( t_s = \frac{3}{\sigma} \) for the specification that \( t_r \leq \frac{6}{10} \) and \( t_s \leq \frac{16}{10} \) we conclude that the requirements translate to \( \omega_n \geq 3 \) and \( \sigma > \frac{15}{8} \). The admissible pole location is the shaded region in the Figure 5 A below and the poles of the system, \(-2 \pm 2\sqrt{3}\) are the two blue dots \( \implies \) constraints are satisfied.

(b) Given that

\[
M_p = \exp \left( \frac{-\zeta \pi}{\sqrt{1 - \zeta^2}} \right) = \exp (-\pi \arctan \theta)
\]

we can compute \( M_p \leq e^{-2} \implies \theta \geq 0.74 \). The new admissible pole locations are shaded blue in Figure 5 B. Since the system poles fall inside the shaded region, the constraints are being met.

(c) Lastly, since \( t_p = \frac{\pi}{\omega_d} \) given \( t_p \leq 1 \) we have that \( \omega_d \geq \pi \). For this, the admissible region is the doubly shaded region in Figure 5 C. Again, the constraints are being satisfied.

Software usage

Every couple of homeworks, software scripts/notebooks from MATLAB/Mathematica will be posted along with the typeset solutions showing how software can be used to solve some of the homework problems. The next few pages show MATLAB and Mathematica code to solve this homework set.

Matlab (Problem 4)

See the attached .mlx file.

Mathematica (Problems 5-8)
Problem 5

Define transfer function and check if it works while solving first question.

\[ G(s) := \frac{4}{s+2}; \]

\[ \text{sinputs} = \{\text{LaplaceTransform}[#, t, s] \} /@ \{2, -1\}; \]

\[ \text{soutputs} = \{G[s]*#\} /@ \text{sinputs}; \]

\[ \{y_1[t], y_2[t]\} = \text{InverseLaplaceTransform}[#, s, t] /@ \text{soutputs} \]

\[ \{4 - 4e^{-2t}, -2 + 2e^{-2t}\} \]

Now solve the second part

\[ \text{soutput} = \text{LaplaceTransform}[\text{Exp}[-3 t], t, s] * G[s] \]

\[ 2 + s \rightarrow \frac{4}{3+s} \]

\[ 4 e^{-3t} (-1 + e^{t}) \]

Problem 6

Define the system parameters

\[ A = \begin{pmatrix} 0 & 1 \\ -k/m & -\rho/m \end{pmatrix}; \]

\[ B = \begin{pmatrix} 0 \\ 1/m \end{pmatrix}; \]

\[ Cc = (1 0); \]

\[ Cn = (c_1 c_2); \]

\[ H[s, Cc_] := \text{First}[Cc.\text{Inverse}[s \cdot \text{IdentityMatrix}[2] - A].B \] Flatten // FullSimplify\]

\[ H[s, Cn] /@ \{c_2 \rightarrow 0, c_1 \rightarrow k\} \]

\[ k \]

Redefine the C matrix that does work and compute the output to a sinusoidal input
\[ C_n = (k \theta); \]

\[
\text{soutput} = H[s, C_n] \ast \text{LaplaceTransform}[\text{Cos}[\omega \ast t], t, s] // \text{FullSimplify}
\]

\[
k s
\]

Perform a partial fraction expansion

\[
\text{Apart}[\text{soutput}, s]
\]

Then spend some time soul-searching on the above expression ... at which point you realize only one term matters in steady state and a big part of it doesn't depend on s.

\[
\text{InverseLaplaceTransform}\left[\frac{\rho \omega^2 + s (k - m \omega^2)}{r\left(s^2 + \omega^2\right)}\right], s, t
\]

Since it is in standard 2nd order form, define the transfer function that way

\[
\text{tf}[s_] := \frac{k/m}{s^2 + \rho / m \ast s + k / m}
\]

\[
\text{assume} = \text{Join}[\ast > 0 & /\{k, m, p\}, \text{Element}[\ast, \text{Reals}] & /\{w, k, m, \rho\}]
\]

Find the magnitude of the response

\[
\text{magGw}[\omega_] := \text{ComplexExpand}\@\text{Abs}[\text{tf}[I \ast w]] // \text{FullSimplify}[\ast, \text{Assumptions} \rightarrow \text{assume}] & /\omega
\]

\[
\text{magGw}[\omega]
\]

Next try to find the phase

\[
\text{FullSimplify}[\text{Arg}@\text{tf}[I \ast w], \text{Assumptions} \rightarrow \text{assume}]
\]

\[
\text{argGw}[\omega] := \text{FullSimplify}\left[\frac{1}{k + w (-m w + i \rho)}\right], \text{Assumptions} \rightarrow \text{assume}
\]

That didn't work so try another track

\[
\text{denTf} = \text{Denominator}@\text{tf}[I \ast w];
\]

\[
\text{argGw}[\omega_] := \text{FullSimplify}\left[\frac{\text{Im}@\text{denTf}}{\text{Re}@\text{denTf}}\right], \text{Assumptions} \rightarrow \text{assume}
\]
argGw[w] /. w \rightarrow \omega

\[ \text{ArcTan} \left[ \frac{\rho \omega}{-k + m \omega^2} \right] \]

With no loss of generality set \( \omega_n = 1 \) via \( m = k = 1/2 \) so that \( \zeta = \rho \).

m = k = 1/2;

Make some plots ...

plots = MapThread[Plot[Evaluate@Table[Tooltip[#1[w], \( \{\rho, 0.1, 1.1, 0.1\} \)],
{w, 0.05, 2}, PlotRange \rightarrow \text{Full}, PlotLabel \rightarrow \#2 &,
{{{magGw, argGw}, ("Magnitude vs. \( \omega \), "Phase vs. \( \omega \))}}];
GraphicsRow[plots]

Problem 7

p7ode = y''[t] + 2 y'[t] + 25 y[t] \[=\] 50;
ics = \{y[0] \rightarrow 0, y'[0] \rightarrow 0\};
p7odeLTS = (LaplaceTransform[p7ode, t, s] /. ics) /. LaplaceTransform[y[t], t, s] \rightarrow Y[s];
p7tf = First@Solve[p7odeLTS, Y[s]] /. Rule \rightarrow (\#2 &)

\[ 25 Y[s] + 2 s Y[s] + s^2 Y[s] = \frac{50}{s} \]

\[ \lim_{s \rightarrow 0} [s \cdot p7tf, s \rightarrow 0] \]

Verify we get the same solution using DSolve as a sanity check

\[ \frac{50}{s (25 + 2 s + s^2)} \]
ys = DSolve[{y''[t] + 2 y'[t] + 25 y[t] == 50, y'[0] == 0, y[0] == 0}, y, t] // First

\[y \rightarrow \text{Function}[\{t\}, \frac{1}{6} e^{-t} \left(12 e^t - 12 \cos(2\sqrt{6} t) - \sqrt{6} \sin(2\sqrt{6} t)\right)]\]

Now make the plot after defining the quantities to mark.

Mp[ζ_] := Exp[-π ζ / \[Sqrt][1 - ζ^2]]
wd[wn_, ζ_] := wn * \[Sqrt][1 - ζ^2]
tp[wd_] := π wd
yp[ys_, z_] := ys (1 + Mp[z])

Plot[y[t] /. ys, {t, 0, 10}, PlotRange -> All, GridLines -> {{0, tp[wd[5, 1/5]]}, {1, yp[2, 1/5]}}, GridLinesStyle -> Directive[Black, Dashed], PlotLabel -> "Y(t) vs. t"]

Problem 8

First define the conditions as constraints:

cond1 = Abs[z] ≥ 3;
cond2 = Re[z] < -30/16;
roots = Solve[s^2 + 6 s + 25 == 0] /. KeyValuePattern[s -> x_] -> X;

Then plot them.

p1[conds_] := ComplexRegionPlot[conds, {z, 2 - 5 I, -10 + 5 I}, GridLines -> {{0}, {0}}, GridLinesStyle -> Directive[Black, Dashed]]; g1 = Show[p1[cond1 && cond2], ComplexListPlot[roots]];

List @@ Reduce[Reduce[Mp[z] ≤ \[Epsilon]^2, (z), \[Reals]] /. z -> Cos[ϕ], (ϕ), \[Reals]]

\[c_1 \in \mathbb{Z}, \quad 2\pi - \text{ArcCos}\left[\frac{2}{\sqrt{4 + \pi^2}}\right] + 2\pi c_1 \leq \phi < 2\pi + 2\pi c_1 \mid 2\pi c_1 < \phi \leq \text{ArcCos}\left[\frac{2}{\sqrt{4 + \pi^2}}\right] + 2\pi c_1\]

Let \(c_1 = 0\) and take the second condition. Then,
2π e_1 < ϕ ≤ ArcCos\left(\frac{2}{\sqrt{4 + \pi^2}}\right) + 2π e_1 /. C[1] \rightarrow 0

Since \( z \) is a complex number below, we specify the condition in terms of its argument.

In[54] :=
cond3 = Abs[Arg[z]] >= \frac{\pi}{\sqrt{4 + \pi^2}};

g2 = Show[p1[cond1 && cond2 && cond3], ComplexListPlot[roots]];