Problem 1. Consider the following second-order differential equation:
\[
\frac{d^2 y}{dt^2} - (4 - y^2) \frac{dy}{dt} + y = 0
\]
(a) Write the dynamics as a non-linear state-space equation.

Remark: State-space models that do not have an input, i.e. are of the form \( \dot{x} = Ax \), are ‘autonomous’, since they evolve on their own.

(b) Identify all equilibria of the system, i.e. points \( x \) such that \( \dot{x} = 0 \). You must both find these equilibria and argue that there are no others.

(c) For each equilibrium point, linearize your dynamics about said equilibrium point, and give the linearized dynamics in state-space form, i.e. \( \dot{x} = Ax \).

Problem 2. Find the transfer function for the block diagram in Figure 1. The answer should be given in terms of the block’s transfer functions, i.e. \( K, G, H, P \).

![Block diagram](image)

Figure 1: The block diagram for Problem 2.

Problem 3. Consider the negative feedback loop in Lecture 5 (Page 15 of the slide). Assume that all state variables, inputs, and outputs are scalars. Assume the model for \( G_1 \) is \( \dot{x}_1 = ax_1 + bu \), \( y = cx_1 + du \). The model for \( G_2 \) is \( \dot{x}_2 = kx_2 + ly \), \( w = mx_2 \).

(a) Find the transfer function from the input \( R \) to the output \( Y \). Your answer should be the ratio of two polynomials in \( s \), with coefficients expressed in terms of \( a, b, c, d, k, l, m \).

(b) Write down the conditions that must be satisfied by \( a, b, c, d, k, l, m \) for this transfer function to be stable, i.e. for all poles of the transfer function to have negative real parts.

Problem 4. Without a computer, determine whether or not the following polynomials have any RHP roots:

(a) \( s^4 + 10s^3 + 40s^2 + 20s + 1 \)

(b) \( s^6 + 2s^5 + 3s^4 + s^3 + s^2 - 3s + 5 \)

(c) \( s^4 + 10s^3 + 10s^2 + 20s + 1 \)

(d) \( s^4 + 10s^3 + 10s^2 + 1 \)
Problem 5. Consider the unity feedback system in Figure 2. Let the plant’s transfer function be given by:

\[ P(s) = \frac{1}{s^3 + 2s^2 + 3s + 1} \]

Suppose our controller is just constant, i.e. \( K(s) = K \). Use the Routh-Hurwitz criterion to determine which values of \( K \) stabilize the closed-loop system.

Problem 6. Consider the six transfer functions given below. For each transfer function, specify the following: (a) Poles, (b) Zeros (if any), (c) Stable or unstable, and (d) Steady-state gain. Use these answers to match each of the six transfer functions with one of the unit step responses in the figure below. All responses were generated with zero initial conditions.

\[ G_1(s) = \frac{-4s + 4}{s^2 + 3s + 4} \]
\[ G_2(s) = \frac{4}{s^2 + 0.3s + 4} \]
\[ G_3(s) = \frac{4}{s^2 + 3s + 4} \]
\[ G_4(s) = \frac{-s + 3}{s - 1} \]
\[ G_5(s) = \frac{300}{s^2 + 101s + 100} \]
\[ G_6(s) = \frac{-3}{s + 1} \]

Problem 7. Consider the following first order system:

\[ \dot{y} = -0.5y + 2u, \quad y(0) = 0 \]  \hspace{1cm} (1)
(a) First, consider a proportional control law \( u(t) = K_p(r(t) - y(t)) \) where \( r(t) \) is the reference command. As mentioned in class, it is typically important, for practical reasons, that \( u(t) \) does not get too large. Consider a unit step command:

\[
 r(t) = \begin{cases} 
 0 & t < 0 \text{ sec} \\
 1 & t \geq 0 \text{ sec} 
\end{cases}
\]  

(2)

For what gains \( K_p \) is \( |u(t)| \leq 1 \) for all time? (Hint: The largest value of \( |u(t)| \) will occur at \( t = 0 \).)

(b) Choose the gain \( K_p \) that satisfies the constraint in part i) and minimizes the steady-state error due to the unit step command. What is the time constant of the closed-loop system for this gain?

(c) Next consider a proportional-integral (PI) control law:

\[
 u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau
\]  

(3)

where \( e(t) = r(t) - y(t) \) is the tracking error. Combine the system model (Equation 1) and PI controller (Equation 3) to obtain a model of the closed-loop system in the form:

\[
 \ddot{y} + a_1 \dot{y} + a_0 y = b_1 \dot{r} + b_0 r
\]  

(4)

How do the damping ratio and natural frequency depend on \( K_p \) and \( K_i \)? What is the steady state error if \( r \) is a unit step?

(d) Keep the value of \( K_p \) designed in part b) and choose \( K_i \) to obtain a damping ratio of \( \zeta = 0.7 \). For these PI gains, what are the estimated maximum overshoot and 5% settling time (neglecting the effect of the zero)?

(e) Plot the output response \( y(t) \) due to a unit step \( r \) for both the P and PI controllers. The closed-loop with the PI controller has a zero due to the term \( b_1 \dot{r} \). Briefly explain how this zero affects the response.

**Problem 8.** Consider the following first order system:

\[
 \ddot{y} - 2 \dot{y} + y = u, \quad y(0) = 0
\]

with a PD controller in the form \( u_t = K_p(r(t) - y(t)) - K_d \dot{y}(t) \).

(a) What is the ODE model for the closed loop from \( r \) to \( y \)?

(b) Choose \( (K_p, K_d) \) so that the closed loop system is stable and has \( (\omega_n, \zeta) = (2, 0.5) \).

(c) What is the steady state error if \( r \) is a unit step reference?

(d) Would you increase or decrease \( K_p \) to reduce the steady state error?