Plan of the Lecture

- **Review**: state-space notions: canonical forms, controllability.
- **Today’s topic**: controllability, stability, and pole-zero cancellations; effect of coordinate transformations; conversion of any controllable system to CCF.

**Goal**: explore the effect of pole-zero cancellations on internal stability; understand the effect of coordinate transformations on the properties of a given state-space model (transfer function; open-loop poles; controllability).

**Reading**: FPE, Chapter 7
State-Space Realizations

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]

- a given transfer function \( G(s) \) can be realized using infinitely many state-space models
- certain properties make some realizations preferable to others
- one such property is *controllability*
Controllability Matrix

Consider a single-input system ($u \in \mathbb{R}$):

$$\dot{x} = Ax + Bu, \quad y = Cx \quad x \in \mathbb{R}^n$$

The Controllability Matrix is defined as

$$C(A, B) = \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix}$$

We say that the above system is **controllable** if its controllability matrix $C(A, B)$ is **invertible**.

- As we will see later, if the system is controllable, then we may assign arbitrary closed-loop poles by *state feedback* of the form $u = -Kx$.
- Whether or not the system is controllable depends on its state-space realization.
Example: Computing $C(A, B)$

Let’s get back to our old friend:

$$
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-6 & -5 \\
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
\end{pmatrix} + \begin{pmatrix}
0 \\
1 \\
\end{pmatrix} u, \quad y = \begin{pmatrix}
1 & 1 \\
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
\end{pmatrix}
$$

Here, $x \in \mathbb{R}^2 \implies A \in \mathbb{R}^{2 \times 2} \implies C(A, B) \in \mathbb{R}^{2 \times 2}$

$$
C(A, B) = [B \mid AB] \quad AB = \begin{pmatrix}
0 & 1 \\
-6 & -5 \\
\end{pmatrix} \begin{pmatrix}
0 \\
1 \\
\end{pmatrix} = \begin{pmatrix}
1 \\
-5 \\
\end{pmatrix}
$$

$$
\implies C(A, B) = \begin{pmatrix}
0 & 1 \\
1 & -5 \\
\end{pmatrix}
$$

Is this system controllable?

$$
\det C = -1 \neq 0 \implies \text{system is controllable}
$$
Controller Canonical Form

A single-input state-space model

\[ \dot{x} = Ax + Bu, \quad y = Cx \]

is said to be in Controller Canonical Form (CCF) is the matrices \( A, B \) are of the form

\[
A = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
* & * & * & \ldots & * & * \\
\end{pmatrix}, \quad B = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
1 \\
\end{pmatrix}
\]

A system in CCF is always controllable!!

(The proof of this for \( n > 2 \) uses the Jordan canonical form, we will not worry about this.)
CCF with Arbitrary Zeros

In our example, we had $G(s) = \frac{s + 1}{s^2 + 5s + 6}$, with a minimum-phase zero at $z = -1$.

Let’s consider a general zero location $s = z$:

$$G(s) = \frac{s - z}{s^2 + 5s + 6}$$

This gives us a CCF realization

$$\begin{align*}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 
\end{bmatrix} &= 
\begin{bmatrix}
0 & 1 \\
-6 & -5 
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 
\end{bmatrix} + 
\begin{bmatrix}
0 \\
1 
\end{bmatrix} u, \\
y &= 
\begin{bmatrix}
-z & 1 
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 
\end{bmatrix}
\end{align*}$$

Since $A, B$ are the same, $\mathcal{C}(A, B)$ is the same $\implies$ the system is still controllable.

A system in CCF is controllable for any locations of the zeros.
OCF with Arbitrary Zeros

Start with the CCF

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-6 & -5
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} + \begin{pmatrix}
0 \\
1
\end{pmatrix} u,
\quad
y = \begin{pmatrix}
-z \\
1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

Convert to OCF: \((A \mapsto A^T, B \mapsto C^T, C \mapsto B^T)\)

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix}
0 & -6 \\
1 & -5
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} + \begin{pmatrix}
-z \\
1
\end{pmatrix} u,
\quad
y = \begin{pmatrix}
0 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

We already know that this system realizes the same t.f. as the original system.

But is it controllable?
OCF with Arbitrary Zeros

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} =
\begin{pmatrix}
0 & -6 \\
1 & -5
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} +
\begin{pmatrix}
-z \\
1
\end{pmatrix} u, \quad y =
\begin{pmatrix}
0 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

\[\bar{A} = A^T, \quad \bar{B} = C^T\]

Let’s find the controllability matrix:

\[
C(\bar{A}, \bar{B}) = [\bar{B} \mid \bar{A}\bar{B}] \quad \bar{A}\bar{B} =
\begin{pmatrix}
0 & -6 \\
1 & -5
\end{pmatrix}
\begin{pmatrix}
-z \\
1
\end{pmatrix} =
\begin{pmatrix}
-6 \\
-z - 5
\end{pmatrix}
\]

\[
\therefore C(\bar{A}, \bar{B}) =
\begin{pmatrix}
-z & -6 \\
1 & -z - 5
\end{pmatrix}
\]

\[
det C = z(z + 5) + 6 = z^2 + 5z + 6 = 0 \quad \text{for } z = -2 \text{ or } z = -3
\]

The OCF realization of the transfer function

\[G(s) = \frac{s - z}{s^2 + 5s + 6}\]

is not controllable when \(z = -2\) or \(-3\), even though the CCF is always controllable.
Beware of Pole-Zero Cancellations!

The OCF realization of the transfer function

\[
G(s) = \frac{s - z}{s^2 + 5s + 6}
\]

is not controllable when \( z = -2 \) or \(-3 \), even though the CCF is always controllable.

Let’s examine \( G(s) \) when \( z = -2 \):

\[
G(s) = \frac{s - z}{s^2 + 5s + 6} \bigg|_{z=-2} = \frac{s+2}{(s+2)(s+3)} = \frac{1}{s+3}
\]

— pole-zero cancellation!

For \( z = -2 \), \( G(s) \) is a first-order transfer function, which can always be realized by this 1st-order controllable model:

\[
\dot{x}_1 = -3x_1 + u, \ y = x_1 \quad \rightarrow \quad G(s) = \frac{1}{s+3}
\]
Beware of Pole-Zero Cancellations!!

We can look at this from another angle: consider the t.f.

\[ G(s) = \frac{1}{s + 3} \]

We can realize it using a one-dimensional controllable state-space model

\[ \dot{x}_1 = -3x_1 + u, \quad y = x_1 \]

or a noncontrollable two-dimensional state-space model

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} =
\begin{pmatrix}
0 & -6 \\
1 & -5
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} +
\begin{pmatrix}
2 \\
1
\end{pmatrix} u,
\quad y = \begin{pmatrix} 0 & 1 \end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

— certainly not the best way to realize a simple t.f.!

Thus, even the *state dimension* of a realization of a given t.f. is not unique!!
Beware of Pole-Zero Cancellations!!

Here is a really bad realization of the t.f.

\[ G(s) = \frac{1}{s + 3}. \]

Use a two-dimensional model:

\[ \dot{x}_1 = -3x_1 + u \]
\[ \dot{x}_2 = 100x_2 \]
\[ y = x_1 \]

► \( x_2 \) is not affected by the input \( u \) (i.e., it is an uncontrollable mode), and not visible from the output \( y \)
► does not change the transfer function
► ... and yet, horrible to implement: \( x_2(t) \propto e^{100t} \)

The transfer function can mask undesirable internal state behavior!!
Pole-Zero Cancellations and Stability

- In case of a pole-zero cancellation, the t.f. contains much less information than the state-space model because some dynamics are “hidden.”
- These dynamics can be either good (stable) or bad (unstable), but we cannot tell from the t.f.
- Our original definition of stability (no RHP poles) is flawed because there can be RHP eigenvalues of the system matrix $A$ that are canceled by zeros, yet they still have dynamics associated with them.

**Definition of Internal Stability (State-Space Version):** A state-space model with matrices $(A, B, C, D)$ is *internally stable* if all eigenvalues of the $A$ matrix are in LHP. This is equivalent to having no RHP open-loop poles and no pole-zero cancellations in RHP.
Now that we have seen that a given transfer function can have many different state-space realizations, we would like a systematic procedure of generating such realizations, preferably with favorable properties (like controllability).

One such procedure is by means of coordinate transformations.
Coordinate Transformations

\[ x \mapsto \bar{x} = Tx, \quad T \in \mathbb{R}^{n \times n} \text{ nonsingular} \]

\[ x = T^{-1} \bar{x} \] (go back and forth between the coordinate systems)
Coordinate Transformations

For example,

\[
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}
\]

This can be represented as

\[
\bar{x} = Tx,
\]

where \( T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \)

The transformation is invertible: \( \det T = -2 \), and

\[
T^{-1} = \frac{1}{\det T} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}
\]

Or we can see this directly:

\[
\bar{x}_1 + \bar{x}_2 = 2x_1; \quad \bar{x}_1 - \bar{x}_2 = 2x_2
\]
Coordinate Transformations and State-Space Models

Consider a state-space model

\[
\dot{x} = Ax + Bu \\
y = Cx
\]

and a change of coordinates \( \bar{x} = Tx \) (\( T \) invertible).

What does the system look like in the new coordinates?

\[
\dot{\bar{x}} = T \dot{x} = T \dot{x} \quad \text{(linearity of derivative)}
\]

\[
= T(Ax + Bu)
\]

\[
= T(AT^{-1} \bar{x} + Bu) \quad \text{(} x = T^{-1} \bar{x} \text{)}
\]

\[
= TAT^{-1} \bar{x} + TB u
\]

\[
y = Cx \quad \text{=} \quad CT^{-1} \bar{x}
\]
Coordinate Transformations and State-Space Models

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]

\[
\begin{align*}
\dot{x} &= A\bar{x} + B\bar{u} \\
y &= C\bar{x}
\end{align*}
\]

where

\[
\bar{A} = TAT^{-1}, \quad \bar{B} = TB, \quad \bar{C} = CT^{-1}
\]

What happens to

- the transfer function?
- the controllability matrix?
Coordinate Transformations and State-Space Models

\[
\dot{x} = Ax + Bu \quad \xrightarrow{T} \quad \dot{x} = \bar{A}\bar{x} + \bar{B}u
\]

\[
y = Cx
\]

where \( \bar{A} = TAT^{-1} \), \( \bar{B} = TB \), \( \bar{C} = CT^{-1} \)

Claim: The transfer function doesn’t change.

Proof:

\[
\bar{G}(s) = \bar{C}(Is - \bar{A})^{-1}\bar{B}
\]

\[
= (CT^{-1}) (Is - TAT^{-1})^{-1} (TB)
\]

\[
= CT^{-1} (TIT^{-1}s - TAT^{-1})^{-1} TB
\]

\[
= CT^{-1} [T (Is - A) T^{-1}]^{-1} TB
\]

\[
= CT^{-1}T (Is - A)^{-1} T^{-1}TB
\]

\[
= C (Is - A)^{-1}B \equiv G(s)
\]
Coordinate Transformations and State-Space Models

\[
\dot{x} = Ax + Bu \quad \xrightarrow{T} \quad \dot{x} = \bar{A}\bar{x} + \bar{B}u
\]

\[
y = Cx
\]

where \( \bar{A} = TAT^{-1} \), \( \bar{B} = TB \), \( \bar{C} = CT^{-1} \)

The transfer function doesn’t change.

In fact:

- open-loop poles don’t change
- characteristic polynomial doesn’t change:

\[
\det(Is - \bar{A}) = \det(Is - TAT^{-1})
\]

\[
= \det[T(Is - A)T^{-1}]
\]

\[
= \det T \cdot \det(Is - A) \cdot \det T^{-1}
\]

\[
= \det(Is - A)
\]
Coordinate Transformations and State-Space Models

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx \\
\end{align*}
\]

where \( \bar{A} = TAT^{-1} \), \( \bar{B} = TB \), \( \bar{C} = CT^{-1} \)

Claim: Controllability doesn’t change.

Proof: For any \( k = 0, 1, \ldots \),

\[
\bar{A}^k \bar{B} = (TAT^{-1})^k TB = TA^k T^{-1}TB = TA^k B \quad \text{(by induction)}
\]

Therefore, \( C(\bar{A}, \bar{B}) = [TB \mid TAB \mid \ldots \mid TA^{n-1}B] \)

\[
= T[B \mid AB \mid \ldots \mid A^{n-1}B]
\]

\[= TC(A, B) \]

Since \( \det T \neq 0 \), \( \det C(\bar{A}, \bar{B}) \neq 0 \) if and only if \( \det C(A, B) \neq 0 \).

Thus, the new system is controllable if and only if the old one is.
Coordinate Transformations and State-Space Models

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]

where \( \bar{A} = TAT^{-1} \), \( \bar{B} = TB \), \( \bar{C} = CT^{-1} \)

Note: The controllability matrix does change:

\[
\begin{align*}
\mathcal{C}(\bar{A}, \bar{B}) &= T \mathcal{C}(A, B) \\
T &= \mathcal{C}(\bar{A}, \bar{B}) \mathcal{C}(A, B)^{-1}
\end{align*}
\]

This is a recipe for going from one controllable realization of a given t.f. to another.

CCF is the most convenient controllable realization of a given t.f., so we want to convert a given controllable system to CCF (useful for control design).
Example: Converting a Controllable System to CCF

Note!! The way I do this is different from the textbook.

Consider $A = \begin{pmatrix} -15 & 8 \\ -15 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ($C$ is immaterial).

Convert to CCF if possible.

Step 1: check for controllability.

$$AB = \begin{pmatrix} -15 & 8 \\ -15 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ -8 \end{pmatrix} \implies C = \begin{pmatrix} 1 & -7 \\ 1 & -8 \end{pmatrix}$$

$\det C = -1$ -- controllable
Example: Converting a Controllable System to CCF

Step 2: Determine desired $C(\bar{A}, \bar{B})$.

We need to figure out $\bar{A}$ and $\bar{B}$.

For CCF, we must have

$$\bar{A} = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so we need to find the coefficients $a_1, a_2$.

Recall: the characteristic polynomial does not change:

$$\det(Is - A) = \det(Is - \bar{A})$$

$$\det \begin{pmatrix} s + 15 & -8 \\ 15 & s - 7 \end{pmatrix} = \det \begin{pmatrix} s & -1 \\ a_2 & s + a_1 \end{pmatrix}$$

$$(s + 15)(s - 7) + 120 = s(s + a_1) + a_2$$

$$s^2 + 8s + 15 = s^2 + a_1 s + a_2$$
Step 2: Determine desired $C(\bar{A}, \bar{B})$.

We need to figure out $\bar{A}$ and $\bar{B}$.

For CCF, we must have

$$\bar{A} = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

We have just computed

$$\bar{A} = \begin{pmatrix} 0 & 1 \\ -15 & -8 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

Therefore, the new controllability matrix should be

$$C(\bar{A}, \bar{B}) = [\bar{B} | \bar{A}\bar{B}] = \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix}.$$
Example: Converting a Controllable System to CCF

Step 3: Compute $T$.

Recall: $T = C(\bar{A}, \bar{B}) \cdot [C(A, B)]^{-1}$

$$C(A, B) = \begin{pmatrix} 1 & -7 \\ 1 & -8 \end{pmatrix}$$

$$[C(A, B)]^{-1} = \begin{pmatrix} 1 & -7 \\ 1 & -8 \end{pmatrix}^{-1} = \frac{1}{-1} \begin{pmatrix} -8 & 7 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & -7 \\ 1 & -1 \end{pmatrix}$$

$$C(\bar{A}, \bar{B}) = \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix}$$

$$T = \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix} \begin{pmatrix} 8 & -7 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$
In the next lecture, we will see why CCF is so useful.