Main Steps

The basic steps of the loopshaping process are:

1) Use a proportional gain to set the desired crossover frequency. This sets the bandwidth / speed of response.
2) Use an integral boost to increase $|L(j\omega)|$ at low frequencies. This improves the reference tracking and disturbance rejection.
3) Use a roll-off to reduce $|L(j\omega)|$ at high frequencies. This improves the noise rejection.
4) Add lead control (if needed) to modify the slope of $|L(j\omega)|$ near the crossover. This is used for closed-loop stability and robustness. This approach can be used on higher-order plants using controllers that are, in general, more complex than a PID controller.
Basic Design Process

Key design parameter: Desired loop crossover $\omega_c$

1. **Proportional Gain**: Select $K_p = \pm \frac{1}{|G(j\omega_c)|}$
   
   Loop $L_1 = G K_p$ has the desired crossover, $|L(j\omega_c)| = 1$.

2. **Integral Boost**: Select $K_i(s) = \frac{s + \omega_i}{s}$ with $\omega_i \leq \omega_c$
   
   Loop $L_2 = G K_p K_i$ has improved low frequency tracking.
   
   Good initial choice $\omega_i = \omega_c/3$ so that $|K_i(j\omega)| \approx 1$ for $\omega \geq \omega_c$.

3. **Roll-off**: Select $K_r(s) = \frac{\omega_r}{s + \omega_r}$ with $\omega_r \geq \omega_c$
   
   Loop $L_3 = G K_p K_i K_r$ has improved noise rejection / robustness.
   
   Good initial choice $\omega_r = 3\omega_c$ so that $|K_r(j\omega)| \approx 1$ for $\omega \leq \omega_c$.

4. **Lead (If needed)**: Select $K_l(s) = \frac{\beta s + \omega_c}{s + \beta \omega_c}$ with $\beta \approx 3 - 10$
   
   Loop $L_4 = G K_p K_i K_r K_l$ has improved stability margins
Example 1: First-Order System

Design a loopshaping controller for $G(s) = -\frac{0.25}{s+0.5}$

Desired crossover at $\omega_c = 1.5 \text{ rad/sec}$
Step 1: Proportional Gain

Plant $G(s) = -\frac{0.25}{s+0.5}$ and desired crossover at $\omega_c = 1.5 \frac{rad}{sec}$

$$K_p = -\frac{1}{|G(j\omega_c)|} = -6.32 \text{ (Note } K_p<0 \text{ because } G(0)<0).$$

$L_1 = G K_p$

![Graph showing magnitude and output response over frequency and time](image)
Step 2: Integral Boost

Plant $G(s) = -\frac{0.25}{s+0.5}$ and desired crossover at $\omega_c = 1.5 \frac{rad}{sec}$

$$K_i = \frac{s+\omega_i}{s} \text{ with } \omega_i = \frac{\omega_c}{3} = 0.5 \frac{rad}{sec}$$

$L_2 = G \cdot K_p \cdot K_i$
Step 3: Roll-off

Plant $G(s) = -\frac{0.25}{s+0.5}$ and desired crossover at $\omega_c = 1.5 \frac{rad}{sec}$

$K_r = \frac{\omega_r}{s+\omega_r}$ with $\omega_r = 3\omega_c = 4.5 \frac{rad}{sec}$

$L_3 = G K_p K_i K_r$
Step 4: Lead

Plant \( G(s) = -\frac{0.25}{s+0.5} \) and desired crossover at \( \omega_c = 1.5 \text{ rad/sec} \)

Loop \( L_3 = G K_p K_i K_r \) has a “shallow” slope near crossover.

The closed-loop is stable with \([0,\infty)\) gain margins and \(\pm 72^\circ\) phase margins.

No lead control is required.

Final Controller:

\[
K(s) = K_p K_i(s) K_r(s) = -\frac{28.5s + 14.2}{s^2 + 4.5s}
\]

\[
\ddot{u}(t) + 4.5\dot{u}(t) = -28.5\dot{e}(t) - 14.2e(t)
\]
Example 1: Matlab Code

\[
\begin{align*}
G &= -\text{tf}(0.25,[1 \ 0.5]); & \quad & \text{\% Plant} \\
wc &= 1.5; & \quad & \text{\% Desired crossover, rad/sec} \\
Kp &= -1/\text{abs}(%evalfr(G, 1j*wc)); & \quad & \text{\% Proportional Gain} \\
wi &= wc/3; & \quad & \text{\% Boost frequency, rad/sec} \\
Ki &= \text{tf}([1 \ wi],[1 \ 0]); & \quad & \text{\% Integral Boost} \\
wr &= 3*wc; & \quad & \text{\% Roll-off frequency, rad/sec} \\
Kr &= \text{tf}(wr,[1 \ wr]); & \quad & \text{\% Roll-off} \\
K &= Kp*Ki*Kr; & \quad & \text{\% Final Controller} \\
L3 &= G*K; & \quad & \text{\% Final loop} \\
S &= \text{feedback}(1,L3); & \quad & \text{\% Closed-loop sensitivity} \\
isstable(S) & \quad & \text{\% Verify closed-loop stability} \\
allmargin(L3) & \quad & \text{\% Classical margins}
\end{align*}
\]
Example 2: Higher-Order System

Design a loopshaping controller for

\[ G(s) = \frac{4}{s^2} \times \frac{400}{s^2 + 0.08s + 400} \times \frac{15}{s + 15} \]

Desired crossover at \( \omega_c = 2.0 \text{ rad/sec} \)

Step 1) Gain: Select

\[ K_p = \frac{1}{|G(j\omega_c)|} \approx 1 \]

Step 2) Boost:

\[ K_i = \frac{s+\omega_i}{s} \]

with \( \omega_i = \frac{\omega_c}{3} \)

Step 3) Rolloff:

\[ K_r = \frac{\omega_r}{s+\omega_r} \]

with \( \omega_r = 3\omega_c \)
Example 2: Higher-Order System

Design a loopshaping controller for

\[ G(s) = \frac{4}{s^2} \frac{400}{s^2 + 0.08s + 400} \frac{15}{s + 15} \]

Desired crossover at \( \omega_c = 2.0 \text{ rad/sec} \)

Step 1) Gain: Select \( K_p = \frac{1}{|G(j\omega_c)|} \approx 1 \)

Step 2) Boost: \( K_i = \frac{s + \omega_i}{s} \)

with \( \omega_i = \frac{\omega_c}{5} \)

Step 3) Rolloff: \( K_r = \frac{\omega_r}{s + \omega_r} \)

with \( \omega_r = 5\omega_c \)
**Step 4: Lead**

Loop \( L_3 = G K_p K_i K_r \) has a “steep” slope near crossover. Closed-loop is unstable with \( L_3 \) so lead control is needed.

\[
K_l(s) = \frac{\beta s + \omega_c}{s + \beta \omega_c} \quad \text{with} \quad \beta = 8
\]

\( L_4 = G K_p K_i K_r K_l \) \( \rightarrow \) Closed-loop is stable 45° of phase margin.
Example 2: Matlab Code

```matlab
>> G1 = tf(1,[1 0 0]);
>> H = 4*tf(400,[1 2*0.02*20 400])*tf(15,[1 15]);
>> G = G1*H; % Plant
>> wc = 2.0; % Desired crossover, rad/sec

>> Kp = 1/abs(evalfr(G, 1j*wc)); % Proportional Gain
>> wi = wc/5; % Boost frequency, rad/sec
>> Ki = tf([1 wi],[1 0]); % Integral Boost
>> wr = 5*wc; % Roll-off frequency, rad/sec
>> Kr = tf(wr,[1 wr]); % Roll-off
>> wl = wc; % Lead frequency, rad/sec
>> beta = 8; % Lead parameter
>> KI = tf([beta wl],[1 beta*wl]); % Lead
>> K = Kp*Ki*Kr*KI; % Final Controller
>> L4 = G*K; % Final loop

>> S = feedback(1,L4); % Closed-loop sensitivity
>> isstable(S) % Verify closed-loop stability
>> allmargin(L4) % Classical margins
```
PID with approximate derivative:

\[ K(s) = K_p + \frac{K_i}{s} + K_d \frac{\alpha s}{s + \alpha} \]

\[ = \frac{(K_p + K_d \alpha) s^2 + (K_p \alpha + K_i) s + K_i \alpha}{s^2 + \alpha s} \]

Loopshaping with proportional, integral boost, and lead:

\[ K(s) = K_p \cdot \frac{s + \bar{\omega}_I}{s} \cdot \frac{\beta s + \omega_c}{s + \beta \omega_c} \]

\[ = \frac{(K_p \beta) s^2 + K_p (\omega_c + \beta \omega_I) s + K_p \omega_I \omega_c}{s^2 + (\beta \omega_c) s} \]

These are different parameterizations for the same class of controllers. Loopshaping can be viewed as a generalization of PID that enables

- Additional controller components (rolloff, notches, etc)
- Closer connection to frequency-domain trade-offs
- Extensions to multivariable systems.
Now we present two important “theorems” regarding the loopshaping design process.

Under mild conditions, the loopshaping design process will yield a stable closed-loop with good stability margins.
Basic Assumptions on $L(s)=G(s)K(s)$

1. $L(s)$ has all poles and zeros in the LHP.
2. $L(0)>0$
3. One crossover $\omega_c$
Basic Assumptions on $L(s)=G(s)K(s)$

1. $L(s)$ has all poles and zeros in the LHP.
2. $L(0)>0$
3. One crossover $\omega_c$
4. Shallow slope ($\geq -30\,\frac{dB}{dec}$) for one decade around $\omega_c$
Basic Assumptions on $L(s) = G(s)K(s)$

1. $L(s)$ has all poles and zeros in the LHP.
2. $L(0) > 0$
3. One crossover $\omega_c$
4. Shallow slope ($\geq -30 \frac{dB}{dec}$) for one decade around $\omega_c$
5. $|L(j\omega)| \geq 2$ for $\omega \leq \omega_1$
Basic Assumptions on $L(s)=G(s)K(s)$

1. $L(s)$ has all poles and zeros in the LHP.
2. $L(0)>0$
3. One crossover $\omega_c$
4. Shallow slope ($\geq -30\frac{dB}{dec}$) for one decade around $\omega_c$
5. $|L(j\omega)| \geq 2$ for $\omega \leq \omega_1$
6. $|L(j\omega)| \leq \frac{1}{2}$ for $\omega \geq \omega_2$
Loopshaping Design Theorem

1. $L(s)$ has all poles and zeros in the LHP.
2. $L(0) > 0$
3. One crossover $\omega_c$
4. Shallow slope ($\geq -30 \frac{dB}{dec}$) for one decade around $\omega_c$
5. $|L(j\omega)| \geq 2$ for $\omega \leq \omega_1$
6. $|L(j\omega)| \leq \frac{1}{2}$ for $\omega \geq \omega_2$

If $L(s)$ satisfies 1-6 then the closed-loop is stable with approximate gain, phase, and disk margins $\geq \pm 6\text{dB}$, $\geq \pm 45^\circ$, and $d_{\text{min}} \geq 0.5$
Loopshaping Design Theorem With Integrators

1. \( L(s) = \frac{1}{sk}H(s) \) where \( H(s) \) has all poles and zeros in the LHP.
2. \( H(0) > 0 \)
3. One crossover \( \omega_c \)
4. Shallow slope (\( \geq -30\, \text{dB/dec} \)) for one decade around \( \omega_c \)
5. \( |L(j\omega)| \geq 2 \) for \( \omega \leq \omega_1 \)
6. \( |L(j\omega)| \leq \frac{1}{2} \) for \( \omega \geq \omega_2 \)

If \( L(s) \) satisfies 1-6 then the closed-loop is stable with approximate gain, phase, and disk margins \( \geq \pm 6\, \text{dB}, \geq \pm 45^\circ \), and \( d_{\text{min}} \geq 0.5 \)
RHP Pole

• Consider a different system that has a pole in the RHP:

\[ G(s) = \frac{10}{s+10} \times \frac{1}{s-9} = \frac{10}{s^2+s-90} \]

• The RHP pole restricts the choice of the loop crossover frequency.
Loopshaping Design Theorem: RHP Pole

1. \( L(s) = \frac{1}{s-p} H(s) \) where \( p>0 \) + \( H(s) \) has all poles and zeros in the LHP.

2. \( H(0)>0 \)

3. One crossover \( \omega_c > 3p \)

4. Shallow slope (\( \geq -30 \text{dB} \text{dec} \)) for one decade around \( \omega_c \)

5. \( |L(j\omega)| \geq 2 \) for \( \omega \leq \omega_1 \)

6. \( |L(j\omega)| \leq \frac{1}{2} \) for \( \omega \geq \omega_2 \)

If \( L(s) \) satisfies 1-6 then the closed-loop is stable with approximate gain, phase, and disk margins \( \geq \pm 6 \text{dB}, \geq \pm 45^\circ \), and \( d_{\text{min}} \geq 0.5 \)
Pre-compensation can be used to filter the reference command and improve the response characteristics.
Overshoot Due To Controller Zeros

Plant: \( G(s) = \frac{1}{s-1} \)

Controller: \( K(s) = \frac{5s+4}{s} \) (Places poles at \( \omega_n = 2 \frac{rad}{sec}, \zeta = 1 \))

Complementary Sensitivity: \( T(s) = \frac{5s+4}{s^2+4s+4} \)

Zero from controller at \( s = -0.8 \frac{rad}{sec} \) appears in \( T(s) \).
Overshoot Due To Controller Zeros

Plant: \( G(s) = \frac{1}{s-1} \)

Controller: \( K(s) = \frac{5s+4}{s} \) (Places poles at \( \omega_n = 2 \text{ rad/sec}, \zeta = 1 \))
Overshoot Due To Controller Zeros

Plant: \( G(s) = \frac{1}{s-1} \)

Controllers: \( K(s) = \frac{K_p s + K_i}{s} \) placing poles at \( \omega_n, \zeta = 1 \).

The overshoot increases as we slow down the speed of response. This is due to a fundamental constraint when controlling unstable plants.
Two Degree-Of-Freedom Controllers

Single Degree-of-Freedom

One type of Two Degree-of-Freedom: Pre-compensation
Pre-compensation

Plant: \( G(s) = \frac{1}{s-1} \)

Controller: \( K(s) = \frac{5s+4}{s} \) (Places poles at \( \omega_n = 2 \text{rad/sec}, \zeta = 1 \))

Complementary Sensitivity: \( T(s) = \frac{5s+4}{s^2+4s+4} \)

Pre-compensation to cancel controller zeros: \( F(s) = \frac{4}{5s+4} \)

\[ \Rightarrow F(s)T(s) = \frac{4}{s^2+4s+4} \]
Pre-compensation:  
\[ F(s) = \frac{4}{5s+4} \implies F(s)T(s) = \frac{4}{s^2+4s+4} \]
Pre-compensation: \( F(s) = \frac{4}{5s+4} \Rightarrow F(s)T(s) = \frac{4}{s^2+4s+4} \)
General Pre-Compensation

Let \( K(s) = \frac{N_K(s)}{D_K(s)} \) where \( N_K(s) \) and \( D_K(s) \) are the numerator and denominator polynomials.

The pre-compensator \( F(s) = \frac{N_K(0)}{N_K(s)} \) will:

- Cancel all zeros in the controller
- Have unity DC gain \( F(0) = 1 \).

The pre-compensator does not have to cancel all zeros in the controller. **Cancelling a RHP zero in the controller will cause** \( F(s) \) **to be unstable and this should never be done.**