ECE 486: Control Systems

Lecture 21: Loopshaping
Frequency-Domain Requirements

Most design requirements can be specified in the frequency domain as bounds:

A) Good reference tracking and disturbance rejection
   \[ |S(j\omega)| \ll 1 \text{ at low frequencies} \]

B) Good noise rejection
   \[ |T(j\omega)| \ll 1 \text{ at high frequencies} \]

C) Reasonable control commands
   \[ |K(j\omega)S(j\omega)| \text{ is bounded} \]

D) Good robustness
   \[ |S(j\omega)| \leq 2.5 \text{ at all frequencies} \]
Requirements: Closed-Loop Stability + Robustness

**Fact:** Closed-loop is stable if and only if all zeros of $1+G(s)K(s)$ are in the LHP.

We require:

A) G(s)K(s) has no pole/zero cancellations in the CRHP

B) $S(s) = \frac{1}{1+G(s)K(s)}$ is stable

We can show that $|S(j\omega)| \leq 2.5$ at all frequencies ensures good disk margins.
**Requirements: Reference Tracking**

**Goal:** The output $y$ should track the reference command $r$. The transfer function from $r$ to $e=r-y$ is:

$$S(s) = \frac{1}{1 + G(s)K(s)} \quad \text{(Sensitivity)}$$

Consider a sinusoidal reference $r(t) = R_0 \cos(\omega t)$. Then:

$$e(t) \rightarrow |S(j\omega)|R_0 \cos(\omega t + \angle S(j\omega))$$

We require $|S(j\omega)| \ll 1$ for good tracking at $\omega$.

If $\omega = 0$ then $r(t) = R_0$ (step) and $e(t) \rightarrow S(j0)R_0$. 

![Control System Diagram](image-url)
**Requirements: Disturbance Rejection**

**Goal:** The disturbance $d$ should have small effect on output $y$. The transfer function from $d$ to $y$ is $G(s)$ in open loop and $G(s)S(s)$ in closed-loop.

Consider a sinusoidal disturbance $d(t) = D_0 \cos(\omega t)$. Then:

(OL) $y(t) \rightarrow |G(j\omega)|D_0 \cos(\omega t + \angle G(j\omega))$

(CL) $y(t) \rightarrow |G(j\omega)S(j\omega)|D_0 \cos(\omega t + \angle G(j\omega)S(j\omega))$

We require $|S(j\omega)| \ll 1$ for good disturbance rejection at $\omega$. 
**Requirements: Noise Rejection**

**Goal:** The noise $n$ should have small effect on output $y$. The transfer function from $n$ to $y$ is $-T(s)$ where:

$$T(s) = \frac{G(s)K(s)}{1+G(s)K(s)} \quad \text{(Complementary Sensitivity)}$$

Consider a sinusoidal noise $n(t) = N_0 \cos(\omega t)$. Then:

$$y(t) \rightarrow -|T(j\omega)|N_0 \cos(\omega t + \angle T(j\omega))$$

We require $|T(j\omega)| \ll 1$ for good noise rejection at $\omega$. 

![Block Diagram](image.png)
**Requirements: Control Effort**

**Goal:** The control $u$ should remain within allowable limits. The transfer function from $r$ to $u$ is $K(s)S(s)$.

Consider a sinusoidal reference $r(t) = R_0 \cos(\omega t)$. Then:

$$u(t) \rightarrow |K(j\omega)S(j\omega)|R_0 \cos(\omega t + \angle K(j\omega)S(j\omega))$$

To remain within saturation limits $|u(t)| \leq u_{\text{max}}$,

$$|K(j\omega)S(j\omega)|R_0 \leq u_{\text{max}} \Rightarrow |K(j\omega)S(j\omega)| \leq \frac{u_{\text{max}}}{R_0}$$

We also need to ensure that $n$ does not cause large $u$. 
Design Requirements: $S(s)$ vs. $T(s)$

Reference tracking and disturbance rejection: $|S(j\omega)| \ll 1$

Noise rejection: $|T(j\omega)| \ll 1$

However $S(s)+T(s)=1$ so we can’t have both $|S(j\omega)| \ll 1$ and $|T(j\omega)| \ll 1$ at the same frequency. This conflict is resolved by splitting the requirements by frequency:

$|S(j\omega)| \ll 1$ at low frequencies and $|T(j\omega)| \ll 1$ at high frequencies.
Basic Frequency Domain Trade-offs

Plant: \( \dot{y}(t) = u(t) \) with \( G(s) = \frac{1}{s} \)

Controller: \( u(t) = K_p e(t) \) with \( K(s) = K_p \)

Loop: \( L(s) = G(s)K(s) = \frac{K_p}{s} \)

Sensitivity: \( S(s) = \frac{1}{1+G(s)K(s)} = \frac{s}{s+K_p} \)

Complementary Sensitivity: \( T(s) = \frac{G(s)K(s)}{1+G(s)K(s)} = \frac{K_p}{s+K_p} \)

Bode magnitude plots for \( K_p = 1 \).
**Basic Frequency Domain Trade-offs**

**Low Frequencies:** Good reference tracking and disturbance rejection but poor noise rejection.

**High Frequencies:** Good noise rejection but poor reference tracking and disturbance rejection.

**Middle Frequencies:** Loop bandwidth $\omega_L$ is where $|L(j\omega_L)| = 1$.

**Loop bandwidth:**

$$L(s) = \frac{K_p}{s} \Rightarrow \omega_L = K_p$$

**Closed-loop time constant:**

$$S(s) = \frac{s}{s+K_p} \Rightarrow \tau = \frac{1}{K_p}$$

Higher bandwidths correspond to faster response.
Basic Frequency Domain Trade-offs

**Low Frequencies:** Good reference tracking and disturbance rejection but poor noise rejection.

**High Frequencies:** Good noise rejection but poor reference tracking and disturbance rejection.

**Middle Frequencies:** Loop bandwidth $\omega_L$ is where $|L(j\omega_L)| = 1$. 

$K_p = 1$
Basic Frequency Domain Trade-offs

**Low Frequencies:** Good reference tracking and disturbance rejection but poor noise rejection.

**High Frequencies:** Good noise rejection but poor reference tracking and disturbance rejection.

**Middle Frequencies:** Loop bandwidth $\omega_L$ is where $|L(j\omega_L)| = 1$. 

$K_p = 10$
Control Effort

Plant: \( \dot{y}(t) = u(t) \) with \( G(s) = \frac{1}{s} \)

Controller: \( u(t) = K_p e(t) \) with \( K(s) = K_p \)

Closed-loop \( r \) to \( u \): \( K(s)S(s) = \frac{K_p s}{s+K_p} \)
Loopshaping is a design method that focuses on the loop $L(s)$. We build the controller from components targeting low, middle, and high frequencies.

**Low Frequencies:** Good reference tracking / disturbance rejection.

\[ |S(j\omega)| \ll 1 \iff |L(j\omega)| \gg 1 \]

**High Frequencies:** Good noise rejection.

\[ |T(j\omega)| \ll 1 \iff |L(j\omega)| \ll 1 \]

**Middle Frequencies (Crossover Region):**

Speed of Response: Loop bandwidth $\omega_L$ such that $|L(j\omega_L)| = 1$

Stability/Robustness: Transition with a shallow slope.
For first- and second-order systems we used settling time and/or rise time as measures of the speed of response. For higher-order systems, an alternative frequency domain notion for speed of response is useful: bandwidth.

1. **Loop Bandwidth, ω_L**: Smallest frequency with \(|L(jω_L)| = 1.0\).

2. **Sensitivity Bandwidth, ω_S**: Highest frequency such that
   \[ |S(jω)| \leq \frac{1}{\sqrt{2}}=-3dB \text{ for all } ω \leq ω_S \]

3. **Complementary Sensitivity Bandwidth, ω_T**: Lowest frequency such that
   \[ |T(jω)| \leq \frac{1}{\sqrt{2}}=-3dB \text{ for all } ω \geq ω_T \]
**Speed of Response: Bandwidth**

Example: \( G(s) = \frac{1}{s} \) and \( K(s) = 12.5 \)

Bandwidths: \( \omega_L = \omega_T = \omega_S = 12.5 \frac{\text{rad}}{\text{sec}} \)

Note that \( S(s) = \frac{s}{s+12.5} \Rightarrow \text{Time Constant } \tau = \frac{1}{12.5 \text{sec}} = \frac{1}{\omega_L} \)
Speed of Response: Bandwidth

Example: \( G(s) = \frac{-0.5s^2 + 1250}{s^3 + 47s^2 + 850s - 3000} \) and \( K(s) = \frac{10s + 30}{s} \)

Bandwidths: \( \omega_S = 5 \frac{rad}{sec}, \omega_L = 12.5 \frac{rad}{sec}, \omega_T = 28 \frac{rad}{sec} \)

Settling Time is \( \approx 0.6 \text{sec} = \frac{3}{\omega_S} \)
Recall $L(s) = G(s)K(s)$, $S(s) = \frac{1}{1+L(s)}$, $T(s) = \frac{L(s)}{1+L(s)}$

**Low Frequencies:** $|S(j\omega)| \ll 1 \iff |L(j\omega)| \gg 1$

Note: $|L(j\omega)| \gg 1 \iff |K(j\omega)S(j\omega)| \approx \frac{1}{|G(j\omega)|}$

**High Frequencies:** $|T(j\omega)| \ll 1 \iff |L(j\omega)| \ll 1$
**Requirements on the Loop $L(s)$**

**Middle Frequencies (Crossover Region):** The slope near $\omega_L$ should not be too steep to ensure stability and robustness.

- A slope of $\approx -40 \frac{dB}{dec}$ means $\angle L(j\omega) \approx -180^\circ$ and closed-loop will be unstable and/or have poor phase margins.

- Slope should not be steeper than $\approx -30 \frac{dB}{dec}$ to ensure $45^\circ$ margin.
Controller Components

Loopshaping builds controllers from the following components:

A) Proportional Gain: A gain (> 1) increases the loop magnitude at all frequencies. This increases bandwidth and reduces steady state error but degrades noise rejection.

B) Integral Boost: Increases the low frequency gain but leaves the high frequencies unchanged. This gives zero steady state error but has negligible effect on bandwidth and noise sensitivity.

C) High Frequency Roll-off: Decreases the high frequency gain but leaves the low frequencies unchanged. This improves noise rejection but has negligible effect on bandwidth and steady-state error.

D) Lead: Makes the slope more shallow near the crossover frequency. This improves robustness but it slightly degrades both the low frequency tracking and high frequency noise rejection.
Example

Plant: \[ \dot{y}(t) + 2y(t) = 5u(t) \] and \[ G(s) = \frac{5}{s+2} \]

Control: \[ K(s) = 1 \]

\[ L(s) = G(s)K(s) = G(s), \quad S(s) = \frac{1}{1+G(s)} = \frac{s+2}{s+7}, \quad T(s) = \frac{G(s)}{1+G(s)} = \frac{5}{s+7} \]
Example

Plant:  \[ \dot{y}(t) + 2y(t) = 5u(t) \quad \text{and} \quad G(s) = \frac{5}{s+2} \]

Control:  \[ K(s) = 1 \]

\[ L(s) = G(s)K(s) = G(s), \quad S(s) = \frac{1}{1+G(s)} = \frac{s+2}{s+7}, \quad T(s) = \frac{G(s)}{1+G(s)} = \frac{5}{s+7} \]

\[ \dot{e}(t) + 7e(t) = \dot{r}(t) + 2r(t) \quad \Rightarrow \text{If } r(t) \rightarrow \bar{r} \text{ then } e(t) \rightarrow S(0)\bar{r} \]

\[ \Rightarrow r(t) \rightarrow 2 \text{ then } e(t) \rightarrow \frac{2}{7} \cdot 2 \approx 0.57 \]
Controller Components

- **Proportional Gain**

- **Integral Boost**

- **Roll-off**

- **Lead**
Proportional Gain

Proportional Gain: $K(s) = K_p$

Recall the following fact for Bode magnitudes in dB:

$$20 \log_{10} |G(j\omega)K(j\omega)| = 20 \log_{10} |G(j\omega)| + 20 \log_{10} |K(j\omega)|$$

Properties:
- If $K_p > 1$ then gain shifts entire loop mag. up.
- If $K_p < 1$ then gain shifts entire loop mag. down.

Proportional gain is used to set the loop bandwidth (crossover frequency).
Effect of Proportional Gain

**Plant:** \[ \dot{y}(t) + 2y(t) = 5u(t) \] and \[ G(s) = \frac{5}{s+2} \]

**Control:** \[ K(s) = 3 = 9.5 \text{dB} \]
Integral Boost

Integral Boost: \( K(s) = \frac{s + \bar{\omega}}{s} \)

Properties:
- Corner frequency \( \bar{\omega} \), high frequency gain \( |K(j\omega)| = 1 \), and low frequency slope of \(-20\frac{dB}{dec}\).
- Corresponds to PI control:
  \[
  \dot{u}(t) = \dot{e}(t) + \bar{\omega} e(t)
  \]
  \[
  \Rightarrow u(t) = e(t) + \bar{\omega} \int_0^t e(\tau) \, d\tau
  \]

Integral boost is used to increase low frequency gain and ensure zero steady-state error.
**Effect of Integral Boost**

**Plant:** \[ \dot{y}(t) + 2y(t) = 5u(t) \] and \[ G(s) = \frac{5}{s+2} \]

**Control:** \[ K(s) = \frac{s + \bar{\omega}}{s} \] with \( \bar{\omega} = 3 \frac{\text{rad}}{\text{sec}} \)

\[ |K(0)| = \infty \Rightarrow |L(0)| = \infty \Rightarrow |S(0)| = \left| \frac{1}{1 + L(0)} \right| = 0 \]

Integral control ensures zero error in steady-state.
High Frequency Roll-off

Roll-off: \( K(s) = \frac{\bar{\omega}}{s + \bar{\omega}} \)

Properties:
- Corner frequency \( \bar{\omega} \), low frequency gain \( |K(j\omega)| = 1 \), and high frequency slope of \(-20 \frac{dB}{dec}\).
- Corresponds to the ODE:
  \[
  \dot{u}(t) + \bar{\omega} u(t) = \bar{\omega} e(t)
  \]

Roll-off is used to decrease high frequency gain and attenuate sensor noise.
Effect of Roll-off

Plant: \[ \dot{y}(t) + 2y(t) = 5u(t) \] and \[ G(s) = \frac{5}{s+2} \]

Control: \[ K(s) = \frac{\bar{\omega}}{s+\bar{\omega}} \text{ with } \bar{\omega} = 10 \frac{\text{rad}}{\text{sec}} \]
Lead: \( K(s) = \frac{\beta s + \bar{\omega}}{s + \beta \bar{\omega}} \)

Properties:

- Zero at \( -\frac{\bar{\omega}}{\beta} \) and pole at \( -\beta \bar{\omega} \),
- Low frequency gain \( \frac{1}{\beta} \) and high frequency gain \( \beta \)
- Positive slope at \( \bar{\omega} \)

Lead is used to make the slope shallower and hence ensure stability and robustness.
Effect of Lead

Plant: \( \ddot{y}(t) = u(t) \) and \( G(s) = \frac{1}{s^2} \)

Control: \( K(s) = \frac{\beta s + \omega}{s + \beta \omega} \) with \( \omega = 1 \frac{\text{rad}}{\text{sec}} \)
The basic steps of the loopshaping process are:

1) Use a proportional gain to set the desired crossover frequency. This sets the bandwidth / speed of response.

2) Use an integral boost to increase $|L(j\omega)|$ at low frequencies. This improves the reference tracking and disturbance rejection.

3) Use a roll-off to reduce $|L(j\omega)|$ at high frequencies. This improves the noise rejection.

4) Add lead control (if needed) to modify the slope of $|L(j\omega)|$ near the crossover. This is used for closed-loop stability and robustness. This approach can be used on higher-order plants using controllers that are, in general, more complex than a PID controller.
**Basic Design Process**

Key design parameter: Desired loop crossover $\omega_c$

1. **Proportional Gain:** Select $K_p = \pm \frac{1}{|G(j\omega_c)|}$

   Loop $L_1 = G K_p$ has the desired crossover, $|L(j\omega_c)| = 1$.

2. **Integral Boost:** Select $K_i(s) = \frac{s+\omega_i}{s}$ with $\omega_i \leq \omega_c$

   Loop $L_2 = G K_p K_i$ has improved low frequency tracking.

   Good initial choice $\omega_i = \omega_c/3$ so that $|K_i(j\omega)| \approx 1$ for $\omega \geq \omega_c$.

3. **Roll-off:** Select $K_r(s) = \frac{\omega_r}{s+\omega_r}$ with $\omega_r \geq \omega_c$

   Loop $L_3 = G K_p K_i K_r$ has improved noise rejection / robustness.

   Good initial choice $\omega_r = 3\omega_c$ so that $|K_r(j\omega)| \approx 1$ for $\omega \leq \omega_c$.

4. **Lead (If needed):** Select $K_l(s) = \frac{\beta s+\omega_c}{s+\beta \omega_c}$ with $\beta \approx 3 - 10$

   Loop $L_4 = G K_p K_i K_r K_l$ has improved stability margins