ECE 486: Control Systems

Lecture 19A: Nyquist Plots, Cauchy’s Argument Principle, Nyquist Stability Condition
Nyquist Plots

A Nyquist plot is a single plot of the frequency response $G(j\omega)$.  
• It consists of the imaginary part $\text{Im}(G(j\omega))$ on the vertical axis versus the real part $\text{Re}(G(j\omega))$ on the horizontal axis.
• The convention is to draw this plot for both $\omega \geq 0$ and $\omega < 0$.

Nyquist plots are used to understand the stability and robustness of a feedback system.

Nyquist plots can be drawn in Matlab using the `nyquist` command. The plots for first order systems (with or without a zero) are simply circles in the complex plane.
Nyquist Plots

Recall that the steady-state sinusoidal response of a stable LTI system is determined by the magnitude and phase of $G(j\omega)$.

$$u(t) = \sin(\omega t) \quad \Rightarrow \quad y(t) \rightarrow |G(j\omega)| \sin(\omega t + \angle G(j\omega))$$

A Bode plot displays $|G(j\omega)|$ and $\angle G(j\omega)$ versus $\omega$ on two separate plots.

A Nyquist plot displays the response $G(j\omega)$ in a different form:

- A single plot of the imaginary part $\text{Im}(G(j\omega))$ vs. $\text{Re}(G(j\omega))$.
- The frequency $\omega$ is implicit on the plot.
- The convention is to draw the plot for both $\omega \geq 0$ and $\omega < 0$. These parts of the curve are complex conjugates.

The Matlab command `nyquist` can be used to draw these plots.
Example

Consider the stable, first-order system:

\[ \dot{y}(t) + 4y(t) = 2u(t) \]

\[ G(s) = \frac{2}{s+4} \]

\[ >> \ G = \text{tf}(2, [1 \ 4]); \]

\[ >> \ \text{bode}(G); \]

\[ >> \ \text{nyquist}(G); \]
Nyquist Plots: First-Order Systems

Consider the stable, first-order system:

\[ \dot{y}(t) + a_0 y(t) = b_0 u(t) \quad G(s) = \frac{b_0}{s + a_0} \]

The real and imaginary parts of the frequency response are:

\[ G(j\omega) = \frac{b_0}{j\omega + a_0} \cdot \frac{-j\omega + a_0}{-j\omega + a_0} = \frac{b_0 a_0}{a_0^2 + \omega^2} + j \frac{-b_0 \omega}{a_0^2 + \omega^2} \]

After some algebra, the real and imaginary parts satisfy:

\[ \left( Re(G(j\omega)) - \frac{b_0}{2a_0} \right)^2 + Im(G(j\omega))^2 = \left( \frac{b_0}{2a_0} \right)^2 \]

This is a circle in the complex plane with center on the real axis at \( \frac{b_0}{2a_0} \) and radius \( \frac{b_0}{2a_0} \).

The Nyquist plot of \( G(s) = \frac{b_1 s + b_0}{s + a_0} \) is also a circle.
Example

Consider the stable, first-order system:

\[ \dot{y}(t) - 4y(t) = 2u(t) \]

\[ G(s) = \frac{2}{s-4} \]

Sketch plot from three points:

- **DC Gain:**
  
  \[ G(0) = -0.5 \]

- **High Frequency:** \( \omega \to \infty \)
  
  \[ G(\omega) \to \frac{2}{j\omega} = -\frac{2j}{\omega} \]

- **Corner Frequency:** \( \omega = 4 \frac{\text{rad}}{\text{sec}} \)
  
  \[ G(4j) \to \frac{2}{4j-4} = -0.25 - 0.25j \]
Example

Consider the stable, first-order system:
\[ \dot{y}(t) - 2y(t) = 3\dot{u}(t) + 5u(t) \quad \text{\(G(s) = \frac{3s+5}{s-2}\)}

Sketch plot from three points:

- **DC Gain:**
  \[ G(0) = -2.5 \]

- **High Frequency:** \( \omega \to \infty \)
  \[ G(\omega) \to 3 \]

- **Corner Frequency:** \( \omega = \frac{2 \text{rad}}{\text{sec}} \)
  \[ G(2j) \to \frac{6j+5}{2j-2} = 0.25 - 2.75j \]
Next, we present a result known as Cauchy’s Argument Principle for a transfer function $G(s)$.

To state the principle:

- Let $\Gamma$ be a simple, closed curve in the complex plane.
- Let $N_p$ and $N_z$ denote the number of poles and zeros of $G(s)$ that lie inside the curve $\Gamma$.

Cauchy’s Argument Principle: $G(s)$ evaluated on the curve $\Gamma$ will encircle the origin $(N_z - N_p)$ times.

This result is used to state a theorem to assess stability of a feedback system using Nyquist plots.
Notation

Let $\Gamma$ be a simple, closed curve in the complex plane:

- Simple: The curve does not intersect itself
- Closed: End point of the curve is the same as the starting point.

$\Gamma_R$ with $R=50$
Notation

Let $\Gamma$ be a simple, closed curve in the complex plane:

- Simple: The curve does not intersect itself
- Closed: End point of the curve is the same as the starting point.

$G(\Gamma)$ denotes the curve obtained by mapping each complex number $s_0 \in \Gamma$ to another complex number $G(s_0)$.

\[
\begin{align*}
G(\Gamma_r) & \text{ with } G(s) = \frac{2}{s+4} \\
\Gamma_r & \text{ with } R=50
\end{align*}
\]
Notation

Let $\Gamma$ be a simple, closed curve in the complex plane:

- Simple: The curve does not intersect itself
- Closed: End point of the curve is the same as the starting point.

$G(\Gamma)$ denotes the curve obtained by mapping each complex number $s_0 \in \Gamma$ to another complex number $G(s_0)$.

$G(\Gamma_R) \to$ Nyquist Plot of $G$ as $R \to \infty$
Cauchy’s Argument Principle

Define:
• $N_p$ := Number of poles of $G(s)$ inside the curve $\Gamma$.
• $N_z$ := Number of zeros of $G(s)$ inside the curve $\Gamma$.

**Principle:** Assume $\Gamma$ does not pass through any poles or zeros of $G(s)$. Then:
• The closed curve $G(\Gamma)$ encircles the origin $N_z - N_p$ times.
• If $N_z - N_p > 0$ then $G(\Gamma)$ encircles the origin clockwise (CW).
• If $N_z - N_p < 0$ then $G(\Gamma)$ encircles the origin counterclockwise (CCW).
Example 1

\( G(s) = s - 1 \) shifts \( \Gamma_R \) to the left by one unit.

- \( N_p := \) Number of poles of \( G(s) \) inside the curve \( \Gamma = 0 \)
- \( N_z := \) Number of zeros of \( G(s) \) inside the curve \( \Gamma = 1 \)

\( \rightarrow G(\Gamma_R) \) encircles the origin \( N_z - N_p = 1 > 0 \) times CW.
Example 2

$G(s) = s + 1$ shifts $\Gamma_R$ to the right by one unit.

- $N_p :=$ Number of poles of $G(s)$ inside the curve $\Gamma = 0$
- $N_z :=$ Number of zeros of $G(s)$ inside the curve $\Gamma = 0$

$\rightarrow G(\Gamma_R)$ encircles the origin $N_z - N_p = 0$ times.
Example 3

\[ G(s) = s^2 - 3s + 2 \] evaluated on \( \Gamma_R \) is a more complicated curve.

- \( N_p := \) Number of poles of \( G(s) \) inside the curve \( \Gamma = 0 \)
- \( N_z := \) Number of zeros of \( G(s) \) inside the curve \( \Gamma = 2 \)

\( \rightarrow G(\Gamma_R) \) encircles the origin \( N_z - N_p = 2 > 0 \) times (CW).
Example 4

\[ G(s) = \frac{2s+4}{s-1} \] evaluated on \( \Gamma_R \) is a more complicated curve.

- \( N_p := \text{Number of poles of } G(s) \text{ inside the curve } \Gamma = 1 \)
- \( N_z := \text{Number of zeros of } G(s) \text{ inside the curve } \Gamma = 0 \)

\[ N_z - N_p = -1 < 0 \text{ times (CCW).} \]
Nyquist Stability Condition

Finally, we cover the Nyquist stability theorem.

Let $L(s) = G(s)K(s)$ be the (open) loop transfer function.

- The value $s = -1$ is a critical point in the Nyquist plot.
- The closed-loop is unstable if the Nyquist curve $L(j\omega_0) = -1$ at some frequency $\omega_0$.

The Nyquist stability theorem states that the closed-loop is stable if and only if the Nyquist curve of $L(s)$ encircles the $s = -1$ point the “correct” number of times. The “correct” number of times is equal to the number of RHP poles of the loop $L(s)$. 
The transfer function \( L(s) = G(s)K(s) \) is called the (open) loop transfer function.

If the Nyquist curve of \( L(s) \) passes through the critical point \( s = -1 \) then the closed-loop is unstable.

- Suppose \( L(j\omega_0) = -1 \) at some frequency \( \omega_0 \). Hence \( 1 + L(j\omega_0) = 0 \).
- The sensitivity \( S(s) = \frac{1}{1+L(s)} \) has a pole on the imaginary axis at \( s = j\omega_0 \).
Example

```matlab
>> G = tf(4, [1 2.0407 4]);
>> K = tf(20, [1 5]);
>> L = G*K;
>> nyquist(L);

>> S = feedback(1, L);
>> pole(S)
ans =
   -7.0407 + 0.0000i
   -0.0000 + 3.7687i
   -0.0000 - 3.7687i

>> evalfr(L, 1j*3.7687)
ans =
   -1.0000 - 0.0000i
```

\[ G(s) = \frac{4}{s^2 + 2.0407s + 4} \]
\[ K(s) = \frac{20}{s + 5} \]
Nyquist Theorem

Notation:

- \( P_{CL} \): Number of poles of the closed-loop in the CRHP.
- \( P_{OL} \): Number of poles of the open-loop \( L(s) \) in the CRHP.
- \( N_{CCW} \): This denotes the number of times the Nyquist curve of \( L(s) \) encircles the critical \(-1\) point. \( N_{CCW} > 0 \) for counterclockwise (CCW) encirclements and \( N_{CCW} < 0 \) for clockwise (CW) encirclements.

**Nyquist Theorem:** Assume \( L(s) = G(s)K(s) \) has no pole/zero cancellations in the CRHP and no poles on the imaginary axis. Then

\[
P_{CL} = P_{OL} - N_{CCW}.
\]

The closed-loop is stable \((P_{CL} = 0)\) if and only if \( N_{CCW} = P_{OL} \).

**Benefit:** Closed-loop stability can be determined from a Nyquist plot of the open loop transfer function \( L(s) \).
**Nyquist Theorem**

The Nyquist theorem follows from Cauchy’s Argument Principle.

- Consider the curve $\Gamma$ given by $\Gamma_R$ as $R \to \infty$. This encloses the RHP and $L(\Gamma)$ is the Nyquist plot of $L(s)$.
- Define $H(s) = 1 + L(s)$. $H(\Gamma)$ encircles the origin $N_z - N_p$ times CW.
- The Nyquist plot $L(\Gamma)$ encircles the -1 point $N_{ccw} = N_p - N_z$ times CCW.
- RHP zeros of $H(s)$ are the RHP poles of closed-loop: $N_z = P_{CL}$.
- RHP poles of $H(s)$ are the RHP poles of $L(s)$: $N_p = P_{OL}$.

Combining these facts:

$$P_{CL} = P_{OL} - N_{ccw}.$$

The theorem can be extended if $L(s)$ has a pole on the imaginary axis.
Example 1

Loop \( L_1(s) = \frac{\frac{2}{s+4}}{s+4} \)

- \( P_{OL} = 0 \)
- \( N_{CCW} = 0 \)

\[ \rightarrow P_{CL} = P_{OL} - N_{CCW} = 0. \]
Closed-loop is stable.

Verify:

\[ S_1(s) = \frac{1}{1 + L_1(s)} \]

\[ = \frac{1}{1 + \frac{\frac{2}{s+4}}{s+4}} = \frac{s + 4}{s + 6} \]
Example 2

Loop $L_2(s) = \frac{-2s+2}{s+4}$

- $P_{OL} = 0$
- $N_{CCW} = -1$

\[ P_{CL} = P_{OL} - N_{CCW} = +1. \]

Closed-loop is unstable.

Verify:

$S_2(s) = \frac{1}{1 + L_2(s)}$

\[ = \frac{1}{1 + \frac{-2s+2}{s+4}} = \frac{s + 4}{-s + 6} \]
Example 3

Loop \( L_3(s) = \frac{2}{s-4} \)

- \( P_{OL} = 1 \)
- \( N_{CCW} = 0 \)

\[ \rightarrow P_{CL} = P_{OL} - N_{CCW} = +1. \]

Closed-loop is unstable.

Verify:

\[ S_3(s) = \frac{1}{1 + L_3(s)} = \frac{1}{1 + \frac{2}{s-4}} = \frac{s - 4}{s - 2} \]
Example 4

Loop \( L_4(s) = \frac{8}{s-4} \)

- \( P_{OL} = 1 \)
- \( N_{CCW} = 1 \)

\[ \rightarrow P_{CL} = P_{OL} - N_{CCW} = 0. \]

Closed-loop is stable.

Verify:

\[ S_4(s) = \frac{1}{1 + L_4(s)} = \frac{1}{1 + \frac{8}{s-4}} = \frac{s-4}{s+4} \]
Example 5

Loop \( L_5(s) = \frac{2}{s-5} \frac{100}{s^2+5s+100} \)

- \( P_{OL} = 1 \)
- \( N_{CCW} = 1 \)

\[ P_{CL} = P_{OL} - N_{CCW} = 0. \]

Closed-loop is stable.