Plan of the Lecture

- Review: control design using frequency response
- Today’s topic: Nyquist stability criterion

**Goal:** learn how to detect the presence of RHP poles of the closed-loop transfer function as the gain $K$ is varied using frequency-response data

**Reading:** FPE, Chapter 6
Review: Frequency Domain Design Method

Design based on Bode plots is good for:

▶ easily visualizing the concepts

▶ evaluating the design and seeing which way to change it

▶ using experimental data (frequency response of the uncontrolled system can be measured experimentally)
Review: Frequency Domain Design Method

Design based on Bode plots is not good for:

- exact closed-loop pole placement (root locus is more suitable for that)
- deciding if a given $K$ is stabilizing or not ...
  - we can only measure how far we are from instability (using GM or PM), if we know that we are stable
  - however, we don’t have a way of checking whether a given $K$ is stabilizing from frequency response data

What we want is a frequency-domain substitute for the Routh–Hurwitz criterion — this is the Nyquist criterion, which we will discuss in today’s lecture.
Nyquist Stability Criterion

Goal: count the number of RHP poles (if any) of the closed-loop transfer function

\[
\frac{KG(s)}{1 + KG(s)}
\]

based on frequency-domain characteristics of the plant transfer function \(G(s)\)
Review: Nyquist Plot

Consider an arbitrary strictly proper transfer function $H$:

$$H(s) = \frac{(s - z_1) \ldots (s - z_m)}{(s - p_1) \ldots (s - p_n)}, \quad m < n$$

Nyquist plot: Im $H(j\omega)$ vs. Re $H(j\omega)$ as $\omega$ varies from $-\infty$ to $\infty$
Nyquist Plot as a Mapping of the $s$-Plane

We can view the Nyquist plot of $H$ as the image of the imaginary axis $\{j\omega : -\infty < \omega < \infty\}$ under the mapping $H : \mathbb{C} \rightarrow \mathbb{C}$
Transformation of a Closed Contour Under $H$

If we choose any closed curve (or contour) $C$ on the left, it will get mapped by $H$ to some other curve (contour) on the right:

Important: when working with contours in the complex plane, always keep track of the direction in which we traverse the contour (clockwise vs. counterclockwise)!!
Phase of $H$ Along a Contour

For any $s \in \mathbb{C}$, the phase (or *argument*) of $H(s)$ is

$$\angle H(s) = \angle \frac{(s - z_1) \ldots (s - z_m)}{(s - p_1) \ldots (s - p_n)}$$

$$= \sum_{i=1}^{m} \angle(s - z_i) - \sum_{j=1}^{n} \angle(s - p_j)$$

$$= \sum_{i=1}^{m} \psi_i - \sum_{j=1}^{n} \varphi_j$$

We are interested in how $\angle H(s)$ changes as $s$ traverses a closed, clockwise (⟳) oriented contour $C$ in the complex plane.

We will look at several cases, depending on how the contour is located relative to poles and zeros of $H$. 
Case 1: Contour Encircles a Zero

Suppose that $C$ is a closed, $\mathcal{O}$-oriented contour in $\mathbb{C}$ that encircles a zero of $H(s)$:

How does $\angle H(s)$ change as we go around $C$?
Case 1: Contour Encircles a Zero

How does $\angle H(s)$ change as we go around $C$?

Let’s see what happens to angles from $s$ to poles/zeros of $H$:

- $\varphi_1$ and $\varphi_2$ return to their original values
- $\psi_1$ picks up a net change of $-360^\circ$
- therefore, $\angle H(s)$ picks up a net change of $-360^\circ$, so $H(C)$ encircles the origin once, clockwise (⟲)
Case 2: Contour Encircles a Pole

Suppose that $C$ is a closed, $⟲$-oriented contour in $\mathbb{C}$ that encircles a pole of $H(s)$:

How does $\angle H(s)$ change as we go around $C$?
Case 2: Contour Encircles a Pole

How does $\angle H(s)$ change as we go around $C$?

Let’s see what happens to angles from $s$ to poles/zeros of $H$:

- $\varphi_1$ and $\psi_1$ return to their original values
- $\varphi_2$ picks up a net change of $-360^\circ$
- therefore, $\angle H(s)$ picks up a net change of $360^\circ$, so $H(C)$ encircles the origin once counterclockwise ($\bigcirc$)
Case 3: Contour Encircles No Poles or Zeros

Suppose that $C$ is a closed, $\mathcal{C}$-oriented contour in $\mathbb{C}$ that does not encircle any poles or zeros of $H(s)$:

How does $\angle H(s)$ change as we go around $C$?
How does $\angle H(s)$ change as we go around $C$?

Let’s see what happens to angles from $s$ to poles/zeros of $H$:

- $\varphi_1, \varphi_2, \psi_1$ all return to their original values
- therefore, no net change in $\angle H(s)$, so $H(C)$ does not encircle the origin
The Argument Principle

These special cases all lead to the following general result:

**The Argument Principle.** Let $C$ be a closed, clockwise $\odot$ oriented contour not passing through any zeros or poles* of $H(s)$. Let $H(C)$ be the image of $C$ under the map $s \mapsto H(s)$:

$$H(C) = \{ H(s) : s \in C \}.$$  

Then:

$$\#(\text{clockwise encirclements } \odot \text{ of } 0 \text{ by } H(C)) = \#(\text{zeros of } H(s) \text{ inside } C) - \#(\text{poles of } H(S) \text{ inside } C).$$

More succinctly,

$$N = Z - P$$

* will see the reason for this later ...
The Argument Principle

\[ N = Z - P \]

- If \( N < 0 \), it means that \( H(C) \) encircles the origin counterclockwise (\( \odot \)).
- We do not want \( C \) to pass through any pole of \( H \) because then \( H(C) \) would not be defined.
- We also do not want \( C \) to pass through any zero of \( H \) because then \( 0 \in H(C) \), so \( \#(\text{encirclements}) \) is not well-defined.
We are interested in RHP poles, so let’s choose a suitable contour $C$ that encloses the RHP:

From now on, $C =$ imaginary axis plus the “path around infinity.”

If $H$ is strictly proper, then $H(\infty) = 0$. 
With this choice of $C$,

$$H(C) = \text{Nyquist plot of } H$$

(image of the imaginary axis under the map $H : \mathbb{C} \rightarrow \mathbb{C}$; if $H$ is strictly proper, $0 = H(\infty)$)
We are interested in RHP roots of $1 + KG(s)$, where $G$ is the plant transfer function.

Thus, we choose $H(s) = 1 + KG(s)$
We now examine the Nyquist plot of $H(s) = 1 + KG(s)$. By the argument principle,

$$N = Z - P,$$

where $N = \#(\bigcirc \text{encirclements of } 0 \text{ by Nyquist plot of } 1 + KG(s))$,

$$Z = \#(\text{zeros of } 1 + KG(s) \text{ inside } C),$$

$$P = \#(\text{poles of } 1 + KG(s) \text{ inside } C)$$

Now we extract information about RHP roots of $1 + KG(s)$
Nyquist Criterion: \( N \)

\[
N = \#(\circlearrowleft \text{ encirclements of } 0 \text{ by Nyquist plot of } 1 + KG(s)) \\
= \#(\circlearrowleft \text{ encirclements of } -1 \text{ by Nyquist plot of } KG(s)) \\
= \#(\circlearrowleft \text{ encirclements of } -1/K \text{ by Nyquist plot of } G(s))
\]

— can be read off the Nyquist plot of the open-loop t.f. \( G \)!!
Nyquist Criterion: \( Z \)

\[
G(s) = \frac{q(s)}{p(s)}, \quad \deg(q) \leq \deg(p)
\]

\[
1 + KG(s) = \frac{p(s) + Kq(s)}{p(s)}
\]

Closed-loop t.f. = \( \frac{KG(s)}{1 + KG(s)} = \frac{Kq(s)}{p(s) + Kq(s)} \)

Therefore:

\[
Z = \#(\text{zeros of } 1 + KG(s) \text{ inside } C)
\]

\[
= \#(\text{RHP zeros of } 1 + KG(s))
\]

\[
= \#(\text{RHP closed-loop poles})
\]
Nyquist Criterion: $P$

\[
G(s) = \frac{q(s)}{p(s)}, \quad \text{deg}(q) \leq \text{deg}(p)
\]

\[
1 + KG(s) = 1 + K \frac{q(s)}{p(s)} = \frac{p(s) + Kq(s)}{p(s)}
\]

Therefore:

\[
P = \#(\text{poles of } 1 + KG(s) \text{ inside } C)
\]

\[
= \#(\text{RHP poles of } 1 + KG(s))
\]

\[
= \#(\text{RHP roots of } p(s))
\]

\[
= \#(\text{RHP open-loop poles})
\]
The Nyquist Theorem

Nyquist Theorem (1928) Assume that \( G(s) \) has no poles on the imaginary axis*, and that its Nyquist plot does not pass through the point \(-1/K\). Then

\[
N = Z - P
\]

\[
\#(\text{of} -1/K \text{ by Nyquist plot of } G(s))
\]

\[
= \#(\text{RHP closed-loop poles}) - \#(\text{RHP open-loop poles})
\]

* Easy to fix: draw an infinitesimally small circular path that goes around the pole and stays in RHP
The Nyquist Stability Criterion

Under the assumptions of the Nyquist theorem, the closed-loop system (at a given gain $K$) is stable if and only if the Nyquist plot of $G(s)$ encircles the point $-1/K$ $P$ times counterclockwise, where $P$ is the number of unstable (RHP) open-loop poles of $G(s)$. 

$$Z = N + P$$

$$Z = 0 \implies N = -P$$
Applying the Nyquist Criterion

Workflow:

\[ \text{Bode } M \text{ and } \phi\text{-plots} \quad \rightarrow \quad \text{Nyquist plot} \]

Advantages of Nyquist over Routh–Hurwitz

- can work directly with experimental frequency response data (e.g., if we have the Bode plot based on measurements, but do not know the transfer function)
- less computational, more geometric (came 55 years after Routh)
**Example**

\[ G(s) = \frac{1}{(s + 1)(s + 2)} \quad \text{(no open-loop RHP poles)} \]

Characteristic equation:

\[ (s + 1)(s + 2) + K = 0 \quad \iff \quad s^2 + 3s + K + 2 = 0 \]

From Routh, we already know that the closed-loop system is stable for \( K > -2 \).

We will now reproduce this answer using the Nyquist criterion.
Example

\[ G(s) = \frac{1}{(s + 1)(s + 2)} \quad \text{(no open-loop RHP poles)} \]

Strategy:

- Start with the Bode plot of \( G \)
- Use the Bode plot to graph \( \text{Im } G(j\omega) \) vs. \( \text{Re } G(j\omega) \) for \( 0 \leq \omega < \infty \)
- This gives only a portion of the entire Nyquist plot

\[ (\text{Re } G(j\omega), \text{Im } G(j\omega)), \quad -\infty < \omega < \infty \]

- Symmetry:

\[ G(-j\omega) = \overline{G(j\omega)} \]

— Nyquist plots are always symmetric w.r.t. the real axis!!
Example

\[ G(s) = \frac{1}{(s + 1)(s + 2)} \]

(no open-loop RHP poles)

Bode plot:

Nyquist plot:
Example: Applying the Nyquist Criterion

\[ G(s) = \frac{1}{(s + 1)(s + 2)} \]  
(no open-loop RHP poles)

Nyquist plot:

\[ \text{\#(\,\circ\, of } -1/K) \]
\[ = \text{\#(RHP CL poles)} - \text{\#(RHP OL poles)} \]
\[ \geq 0 \]

\[ \implies K \in \mathbb{R} \text{ is stabilizing if and only if } \text{\#(\,\circ\, of } -1/K) = 0 \]

- If \( K > 0 \), \( \text{\#(\,\circ\, of } -1/K) = 0 \)
- If \( 0 < -1/K < 1/2 \), \( \text{\#(\,\circ\, of } -1/K) > 0 \implies \text{closed-loop stable for } K > -2 \)