Plan of the Lecture

- **Review**: control design using frequency response: PI/lead
- **Today’s topic**: control design using frequency response: PD/lag, PID/lead+lag

**Goal**: understand the effect of various types of controllers (PD/lead, PI/lag) on the closed-loop performance by reading the open-loop Bode plot; develop frequency-response techniques for shaping transient and steady-state response using dynamic compensation

**Reading**: FPE, Chapter 6
Review: Bode’s Gain-Phase Relationship

Assuming that $G(s)$ is minimum-phase (i.e., has no RHP zeros), we derived the following for the Bode plot of $KG(s)$:

<table>
<thead>
<tr>
<th></th>
<th>low freq.</th>
<th>real zero/pole</th>
<th>complex zero/pole</th>
</tr>
</thead>
<tbody>
<tr>
<td>mag. slope</td>
<td>$n$</td>
<td>up/down by 1</td>
<td>up/down by 2</td>
</tr>
<tr>
<td>phase</td>
<td>$n \times 90^\circ$</td>
<td>up/down by $90^\circ$</td>
<td>up/down by $180^\circ$</td>
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</table>

We can state this succinctly as follows:

**Gain-Phase Relationship.** Far enough from break-points,

$$\text{Phase} \approx \text{Magnitude Slope} \times 90^\circ$$
Bode’s Gain-Phase Relationship

Gain-Phase Relationship. Far enough from break-points,

\[ \text{Phase} \approx \text{Magnitude Slope} \times 90^\circ \]

This suggests the following rule of thumb:

- \( M \) has slope \(-2\) at \( \omega_c \)
  \[ \Rightarrow \phi(\omega_c) = -180^\circ \]
  \[ \Rightarrow \text{bad (no PM)} \]

- \( M \) has slope \(-1\) at \( \omega_c \)
  \[ \Rightarrow \phi(\omega_c) = -90^\circ \]
  \[ \Rightarrow \text{good (PM = } 90^\circ) \]

— this is an important design guideline!!

(Similar considerations apply when \( M \)-plot has positive slope – depends on the t.f.)
Bode’s Gain-Phase Relationship suggests that we can shape the time response of the *closed-loop* system by choosing $K$ (or, more generally, a dynamic controller $KD(s)$) to tune the Phase Margin.

In particular, from the quantitative Gain-Phase Relationship,

$$\text{Magnitude slope}(\omega_c) = -1 \quad \Rightarrow \quad \text{Phase}(\omega_c) \approx -90^\circ$$

— which gives us PM of $90^\circ$ and consequently good damping.
Lead Controller Design Using Frequency Response
General Procedure

1. Choose $K$ to get desired bandwidth spec w/o lead
2. Choose lead zero and pole to get desired PM
   ▶ in general, we should first check PM with the $K$ from 1, w/o lead, to see how much more PM we need
3. Check design and iterate until specs are met.

This is an intuitive procedure, but it’s not very precise, requires trial & error.
Lag Compensation: Bode Plot

\[ D(s) = \frac{s + z}{s + p} = \frac{z}{p} \frac{s}{s + p} + 1, \quad z \gg p \]

\[ \frac{j\omega + z}{j\omega + p} \xrightarrow{\omega \to \infty} 1 \]

so \( M \to 1 \) at high frequencies

- subtracts phase, hence the term “phase lag”
Lag Compensation: Bode Plot

\[ \frac{j\omega + z}{j\omega + p} \xrightarrow{\omega \to 0} \frac{z}{p} \]

steady-state tracking error:

\[ e(\infty) = \frac{sR(s)}{1 + D(s)G(s)} \bigg|_{s=0} \]

large \( z/p \) \( \Rightarrow \) better s.s. tracking

- lag decreases \( \omega_c \) \( \Rightarrow \) slows down time response (to compensate, adjust \( K \) or add lead)

- caution: lead increases PM, but adding lag can undo this

- to mitigate this, choose both \( z \) and \( p \) very small, while maintaining desired ratio \( z/p \)
Example

\[ G(s) = \frac{1}{(s + 0.2)(s + 0.5)} \quad \text{Bode form} \quad \equiv \quad \frac{10}{(s/0.2 + 1)(s/0.5 + 1)} \]

Objectives:

- PM \( \geq 60^\circ \)
- \( e(\infty) \leq 10\% \) for constant reference (closed-loop tracking error)

Strategy:

- we will use lag

\[ KD(s) = K \frac{s + z}{s + p}, \quad z \gg p \]

- \( z \) and \( p \) will be chosen to get good tracking
- PM will be shaped by choosing \( K \)
- this is different from what we did for lead (used \( p \) and \( z \) to shape PM, then chose \( K \) to get desired bandwidth spec)
Step 1: Choose $K$ to Shape PM

Check Bode plot of $G(s)$ to see how much PM it already has:

- from Matlab, $\omega_c \approx 1$
- PM $\approx 40^\circ$
- we want PM $= 60^\circ$

\[
\phi = -120^\circ \quad \text{at } \omega \approx 0.573
\]
\[
M = 2.16
\]

— need to decrease $K$ to $1/2.16$

A conservative choice (to allow some slack) is $K = 1/2.5 = 0.4$, gives $\omega_c \approx 0.52$, PM $\approx 65^\circ$
Step 2: Choose $z$ & $p$ to Shape Tracking Error

So far: $KG(s) = \frac{0.4 \cdot 10}{\left(\frac{s}{0.2} + 1\right) \left(\frac{s}{0.5} + 1\right)}$

$$e(\infty) = \frac{1}{1 + KG(s)} \bigg|_{s=0} = \frac{1}{1 + 4} = \frac{1}{5} = 20\% \quad \text{(too high)}$$

To have $e(\infty) \leq 10\%$, need $KD(0)G(0) \geq 9$:

$$e(\infty) = \frac{1}{1 + KD(0)G(0)} \leq \frac{1}{1 + 9} = 10\%.$$  

So, we need

$$D(0) = \frac{s + z}{s + p} \bigg|_{s=0} = \frac{z}{p} \geq \frac{9}{4} = 2.25 \quad \text{— say, } z/p = 2.5$$

Not to distort PM and $\omega_c$, let’s pick $z$ and $p$ an order of magnitude smaller than $\omega_c \approx 0.5$: $z = 0.05$, $p = 0.02$
Overall Design

Plant:
\[ G(s) = \frac{10}{\left( \frac{s}{0.2} + 1 \right) \left( \frac{s}{0.5} + 1 \right)} \]

Controller:
\[ KD(s) = 0.4 \frac{s + 0.05}{s + 0.02} \]

--- the design still needs a bit of refinement ...
Lead & Lag Compensation

Let’s combine the advantages of PD/lead and PI/lag.

Back to our example: \[ G(s) = \frac{10}{\left(\frac{s}{0.2} + 1\right) \left(\frac{s}{0.5} + 1\right)} \]

- from Matlab, \( \omega_c \approx 1 \)
- PM \( \approx 40^\circ \)

New objectives:
- \( \omega_{BW} \geq 2 \)
- PM \( \geq 60^\circ \)
- \( e(\infty) \leq 1\% \) for const. ref.
What we got before, with lag only:

- Improved PM by adjusting $K$ to decrease $\omega_c$.
- This gave $\omega_c \approx 0.5$, whereas now we want a larger $\omega_c$
  (recall: $\omega_{BW} \in [\omega_c, 2\omega_c]$, so $\omega_c = 0.5$ is too small)

So: we need to reshape the phase curve using lead.
Step 1. Choose $K$ to get $\omega_c \approx 2$ (before lead)

Using Matlab, can check:

at $\omega = 2$, $M \approx 0.24$ (with $K = 1$)

— need $K = \frac{1}{0.24} \approx 4.1667$

— choose $K = 4$

(gives $\omega_c$ slightly $< 2$, but still ok).
Lead & Lag Compensation

\[ K = 4 \]

Step 2. Decide how much phase lead is needed, and choose \( z_{\text{lead}} \) and \( p_{\text{lead}} \)

Using Matlab, can check:

\[ \text{at } \omega = 2, \quad \phi \approx -160^\circ \]

— so PM = 20°

(in fact, choosing \( K = 4 \) made things worse: it increased \( \omega_c \) and consequently decreased PM)

We need at least 40° phase lead!!

The choice of lead pole/zero must satisfy

\[ \sqrt{z_{\text{lead}} \cdot p_{\text{lead}}} \approx 2 \quad \Rightarrow \quad z_{\text{lead}} \cdot p_{\text{lead}} = 4 \]
Lead & Lag Compensation

Need at least 40° phase lead, while satisfying

\[ \sqrt{z_{\text{lead}} \cdot p_{\text{lead}}} \approx 2 \implies z_{\text{lead}} \cdot p_{\text{lead}} = 4 \]

Let’s try \( z_{\text{lead}} = 1 \) and \( p_{\text{lead}} = 4 \)

\[ D(s) = \frac{s + 1}{\frac{s}{4} + 1} \]

Phase lead = 37° — not enough!!
Lead & Lag Compensation

Need at least 40° phase lead, while satisfying

$$\sqrt{z_{lead} \cdot p_{lead}} \approx 2 \implies z_{lead} \cdot p_{lead} = 4$$

The choice of $z_{lead} = 1$, $p_{lead} = 4$ gave phase lead $= 37°$.

Need to space $z_{lead}$ and $p_{lead}$ farther apart:

$$\begin{cases} 
    z_{lead} = 0.8 \\
    p_{lead} = 5 
\end{cases} \implies \text{phase lead} = 46°$$
Lead & Lag Compensation

Step 3. Evaluate steady-state tracking and choose \( z_{\text{lag}}, p_{\text{lag}} \) to satisfy specs

So far:

\[
K D(s) G(s) = 4 \left( \frac{s}{0.8} + 1 \right) \left( \frac{s}{0.5} + 1 \right) \left( \frac{s}{0.2} + 1 \right) \cdot \frac{10}{\left( \frac{s}{5} + 1 \right)}
\]

\[
K D(0) G(0) = 40 \quad \implies \quad e(\infty) = \frac{1}{1 + K D(0) G(0)} = \frac{1}{1 + 40}
\]

— this is not small enough: need \( 1\% = \frac{1}{100} = \frac{1}{1 + 99} \)

We want \( D(0) \geq \frac{99}{40} \) with lag \( \frac{z_{\text{lag}}}{p_{\text{lag}}} \approx 2.5 \) will do
Lead & Lag Compensation

Need to choose lag pole/zero that are sufficiently small (not to distort the phase lead too much) and satisfy \( \frac{z_{\text{lag}}}{p_{\text{lag}}} \approx 2.5 \).

We can stick with our previous design:

\[ z_{\text{lag}} = 0.05, \quad p_{\text{lag}} = 0.02 \]

Overall controller:

\[
\frac{s}{4 \frac{0.8}{s} + 1} \cdot \frac{s + 0.05}{s + 0.02}
\]

\( \text{lead (with gain } K = 4 \text{ absorbed)} \)

\( \text{lag (not in Bode form)} \)

(Note: we don’t rewrite lag in Bode form, because \( z_{\text{lag}}/p_{\text{lag}} \) is not incorporated into \( K \).)
Frequency Domain Design Method: Advantages

Design based on Bode plots is good for:

- easily visualizing the concepts
- evaluating the design and seeing which way to change it
- using experimental data (frequency response of the uncontrolled system can be measured experimentally)
Frequency Domain Design Method: Disadvantages

Design based on Bode plots is not good for:

- exact closed-loop pole placement (root locus is more suitable for that)
- deciding if a given $K$ is stabilizing or not ...
  - we can only measure \textit{how far} we are from instability (using GM or PM), if we know that we are stable
  - however, we don’t have a way of checking whether a given $K$ is stabilizing from frequency response data

What we want is a frequency-domain substitute for the Routh–Hurwitz criterion — this is the Nyquist criterion, which we will discuss in the next lecture.