ECE 486 (Control Systems) – Homework 2

Due: Feb. 6

Review of Laplace transforms

Problem 1. *Calculate the following Laplace transforms* $F_i = \mathcal{L}\{f_i\}$ **by hand***:*

i) $f_1(t) = \sin(t)$

Hint: *Recall Euler's formula.*

ii) $f_2(t) = e^{-5t}$

iii) $f_3(t) = \sin(t) + e^{-5t}$

Calculate $\lim_{t\to\infty} f_i(t)$*, if it exists, for the 3 previous functions. For which functions can the consequent of the Final Value Theorem be used?*

Step responses

Problem 2. *Compute the step responses of the following transfer functions* **by hand***:*

i) $H_1(s) = \frac{4}{s+20}$

ii) $H_2(s) = \frac{4}{s-20}$

Recall that a step response is the output of the system when all the initial conditions are zero and input is as follows:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Calculate the steady-state value of each step response.

Problem 3. *Consider the following transfer functions:*

i) $H_1(s) = \frac{1}{s^2-s+2}$

ii) $H_2(s) = \frac{s-3}{s^2+5s+6}$

Calculate $\lim_{s\to 0} H_i(s)$*. Use the MATLAB command* `step` *to plot their step responses, and attach the plots. (The command* `ltiview` *may also be useful.) Can the Final Value Theorem be invoked? What is the DC gain?*

Nonlinear models and linearization

Problem 4. *Figure 1 depicts an inverted pendulum mounted onto a cart. The angle the pendulum makes is denoted* θ*, the length of the pendulum is* ℓ*, and the pendulum itself is modeled as a single point mass at the end with mass* m*. A force* F *is applied to the side of the cart in the horizontal direction. The angle* θ *and force* F *are related through the following ordinary differential equation (ODE):*

$$\ddot{\theta} + \frac{\gamma}{J}\dot{\theta} - \frac{mg\ell}{J}\sin(\theta) - \frac{l}{J}F\cos(\theta) = 0$$

Here, γ *is the coefficient of viscous friction,* J *is the total angular momentum of the system, and* g *is the acceleration due to gravity. You may take this ODE as given; you do* **not** *need to derive it.*

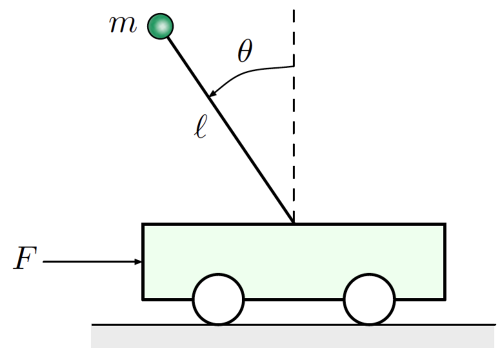

Figure 1: An inverted pendulum attached to a cart; this is typically referred to as the 'cart-pole' model.

i) *Write down the nonlinear state-space model for the cart-pole system with input* $u = F$ *and output* $y = \theta$.

ii) *Show that the zero-state (*$\theta(0) = 0$, $\dot{\theta}(0) = 0$*) and zero-input (*$u(t) = 0$ *for all* $t \geq 0$*) is an equilibrium point and linearize the system around it.*

iii) *Derive the transfer function* $H(s)$ *from the linearized state-space model.*

Hint: *Write out the differential equations in the time domain, switch to the s-domain using the differentiation rule for Laplace transforms, and use the resulting equations to solve for* $Y(s)$ *in terms of* $U(s)$.

Transfer functions and state-space models

Problem 5. *The following state-space dynamics are in* controller-canonical form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 & b_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Write out the differential equations, convert them to the s-domain, and find the transfer function mapping $U(s)$ *to* $Y(s)$.

Repeat this for observer-canonical form*:*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -a_0 \\ 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Can two different state-space models give rise to the same transfer function?