

Plan of the Lecture

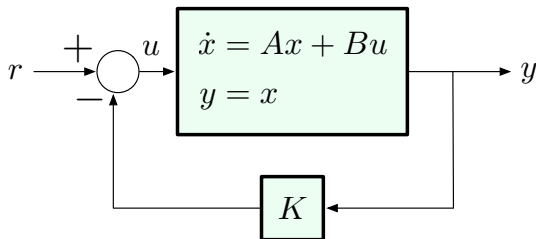
- ▶ **Review:** arbitrary pole placement by full state feedback.
- ▶ **Today's topic:** observer design for state estimation when full state feedback is not implementable.

Goal: for **observable** systems (definition to be introduced today), learn how to estimate the state x from output $y = Cx$ using an observer.

Reading: FPE, Chapter 7

Review: Pole Placement via State Feedback

Assume that the plant is controllable:



$$\dot{x} = Ax + B(-Kx + r) = (A - BK)x + Br, \quad y = x$$

Transfer function from R to Y :

$$Y(s) = (Is - A + BK)^{-1}BR(s)$$

Closed-loop poles are the eigenvalues of $A - BK$!!

Review: Pole Placement in CCF

$$\dot{x} = (A - BK)x + Br, \quad y = Cx$$

$$A - BK = - \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ a_n + k_1 & a_{n-1} + k_2 & \dots & a_2 + k_{n-1} & a_1 + k_n \end{pmatrix}$$

Closed-loop poles are the roots of the characteristic polynomial

$$\begin{aligned} \det(Is - A + BK) \\ = s^n + (a_1 + k_n)s^{n-1} + \dots + (a_{n-1} + k_2)s + (a_n + k_1) \end{aligned}$$

Key observation: When the system is in CCF, each control gain affects only *one* of the coefficients of the characteristic polynomial, and these coefficients can be assigned arbitrarily by a suitable choice of k_1, \dots, k_n .

Hence the name **Controller Canonical Form** — convenient for control design.

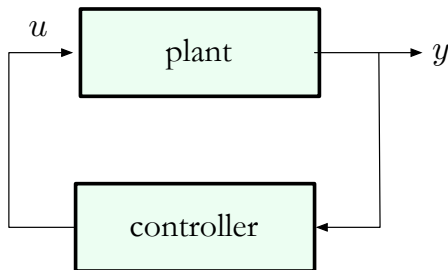
Pole Placement by State Feedback

General procedure for any *controllable* system:

1. Convert to CCF using a suitable invertible coordinate transformation T (such a transformation exists by controllability).
2. Solve the pole placement problem in the new coordinates.
3. Convert back to original coordinates.

Is Full State Feedback Always Available?

In a typical system, measurements are provided by sensors:



Full state feedback $u = -Kx$ is *not implementable!!*

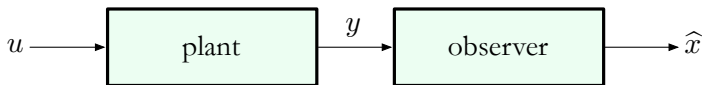
When Full State Feedback Is Unavailable ...

... we need an **Observer!!**



State Estimation Using an Observer

When full state feedback is unavailable, the **observer** is used to **estimate** the state x :

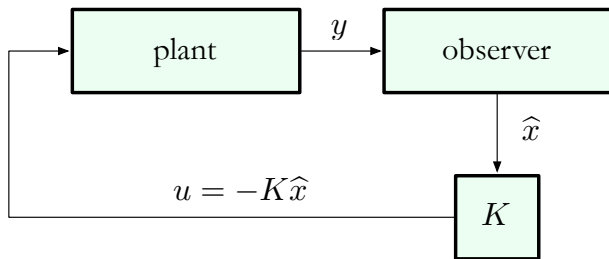


State Estimation Using an Observer

The idea is to design the observer in such a way that the state estimate \hat{x} is *asymptotically accurate*:

$$\|\hat{x}(t) - x(t)\| = \sqrt{\sum_{i=1}^n (\hat{x}_i(t) - x_i(t))^2} \xrightarrow{t \rightarrow \infty} 0$$

If we are successful, then we can try **estimated state feedback**:



A New Concept: Observability

- ▶ Before, we saw that closed-loop poles can be assigned arbitrarily by full state feedback when the plant is **controllable**.
- ▶ Now, we will see that asymptotically accurate state estimation will be possible when the system is **observable**.
- ▶ **Observability** is a system property which is dual to **controllability**.

Observability

Consider a single-output system ($y \in \mathbb{R}$):

$$\dot{x} = Ax + Bu, \quad y = Cx \quad x \in \mathbb{R}^n$$

The **Observability Matrix** is defined as

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- recall that C is $1 \times n$ and A is $n \times n$, so $\mathcal{O}(A, C)$ is $n \times n$;
- the observability matrix only involves A and C , not B

We say that the above system is **observable** if its observability matrix $\mathcal{O}(A, C)$ is *invertible*.

(This definition is only true for the single-output case; the multiple-output case involves the *rank* of $\mathcal{O}(A, C)$.)

Example: Computing $\mathcal{O}(A, C)$

$$\text{Let } A = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

Here, $n = 2$, $C \in \mathbb{R}^{1 \times 2}$, $A \in \mathbb{R}^{2 \times 2} \implies \mathcal{O}(A, C) \in \mathbb{R}^{2 \times 2}$.

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \end{bmatrix}$$

$$\text{where } CA = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix} = \begin{pmatrix} 1 & -5 \end{pmatrix}$$

$$\therefore \mathcal{O}(A, C) = \begin{pmatrix} 0 & 1 \\ 1 & -5 \end{pmatrix}$$

$$\det \mathcal{O}(A, C) = -1 \implies \text{the system is observable}$$

— recall: this system is in **Observer Canonical Form (OCF)** ...

Observer Canonical Form

A single-output state-space model

$$\dot{x} = Ax + Bu, \quad y = Cx$$

is said to be in **Observer Canonical Form** (OCF) if the matrices A, C are of the form

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & * \\ 1 & 0 & \dots & 0 & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & * \\ 0 & 0 & \dots & 0 & 1 & * \end{pmatrix}, \quad C = (0 \quad 0 \quad \dots \quad 0 \quad 1)$$

Fact: A system in OCF is *always observable*!!

(The proof of this for $n > 2$ uses the Jordan canonical form, we will not worry about this.)

Coordinate Transformations and Observability

Just like controllability, observability is preserved under invertible coordinate transformations.

$$\begin{array}{ccc} \dot{x} = Ax + Bu & \xrightarrow{T} & \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = Cx & & y = \bar{C}\bar{x} \end{array}$$

$$\text{where } \bar{A} = TAT^{-1}, \quad \bar{B} = TB, \quad \bar{C} = CT^{-1}$$

$$\begin{aligned} \mathcal{O}(\bar{A}, \bar{C}) &= \begin{pmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \\ \bar{C}\bar{A}^{n-1} \end{pmatrix} = \begin{pmatrix} CT^{-1} \\ CT^{-1}TAT^{-1} \\ \vdots \\ CT^{-1}TA^{n-1}T^{-1} \end{pmatrix} \\ &= \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} T^{-1} = \mathcal{O}(A, C)T^{-1} \end{aligned}$$

Coordinate Transformations and Observability

Just like controllability, observability is preserved under invertible coordinate transformations:

$$\begin{array}{ccc} \dot{x} = Ax + Bu & \xrightarrow{T} & \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = Cx & & y = \bar{C}\bar{x} \end{array}$$

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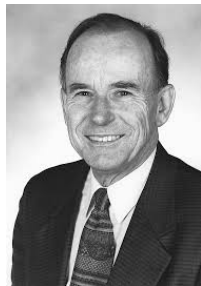
If the original system is observable, then

$$\begin{array}{c} T \underbrace{[\mathcal{O}(A, C)]^{-1}}_{\text{old}} = \underbrace{[\mathcal{O}(\bar{A}, \bar{C})]^{-1}}_{\text{new}} \\ \Downarrow \\ T = \underbrace{[\mathcal{O}(\bar{A}, \bar{C})]^{-1}}_{\text{new}} \underbrace{[\mathcal{O}(A, C)]}_{\text{old}} \end{array}$$

Observability and State Estimation

As we will show next:

If the system is observable, then there exists an observer (state estimator) that provides an asymptotically convergent estimate \hat{x} of the state x based on the observed output y .



The particular type of observer we will construct is called the **Luenberger observer** after David G. Luenberger, who developed this idea in his 1963 Ph.D. dissertation.

David Luenberger is a Professor at Stanford University.

The Luenberger Observer

Consider a state-space model

$$\dot{x} = Ax \quad (\text{for now, assume } u = 0)$$

$$y = Cx$$

We wish to estimate the state x based on the output y .

Consider feeding the output y as input to the following system with state \hat{x} :

$$\dot{\hat{x}} = (A - LC)\hat{x} + Ly.$$

Assumption: The **output injection matrix** L is chosen in such a way that the matrix $A - LC$ is **Hurwitz** (i.e., all of its eigenvalues lie in LHP).

At this point, we do not assume anything about observability.

The Luenberger Observer

$$\text{System:} \quad \dot{x} = Ax$$

$$y = Cx$$

$$\text{Observer:} \quad \dot{\hat{x}} = (A - LC)\hat{x} + Ly.$$

What happens to **state estimation error** $e = x - \hat{x}$ as $t \rightarrow \infty$?

$$\begin{aligned}\dot{e} &= \dot{x} - \dot{\hat{x}} \\ &= Ax - [(A - LC)\hat{x} + LCx] \\ &= (A - LC)x - (A - LC)\hat{x} \\ &= (A - LC)e\end{aligned}$$

Does $e(t)$ converge to zero in some sense?

Linear ODEs and Eigenvalues: A Digression

$$\dot{v} = Fv, \quad v \in \mathbb{R}^n, \quad F \in \mathbb{R}^{n \times n}$$

Let $\lambda_1, \dots, \lambda_n$ be eigenvalues of F , i.e., roots of $\det(Is - F) = 0$.

Then there exists a matrix $T \in \mathbb{R}^{n \times n}$, such that $T^{-1} = T^T$ and

$$F = T^{-1} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} T$$

Consider the change of coordinates $\bar{v} = Tv$. Then

$$\dot{\bar{v}} = T F T^{-1} \bar{v} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \bar{v}$$

Linear ODEs: A Digression

$$\dot{\bar{v}} = T F T^{-1} \bar{v} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \bar{v}, \quad (\lambda_1, \dots, \lambda_n) = \text{eig}(F)$$



$$\dot{\bar{v}}_i = \lambda_i \bar{v}_i, \quad i = 1, 2, \dots, n$$

This system of n 1st-order ODEs has the solution

$$\bar{v}_i(t) = \bar{v}_i(0) e^{\lambda_i t}, \quad i = 1, 2, \dots, n$$

If all λ_i 's have negative real parts, then

$$\begin{aligned} \|v(t)\|^2 &= v(t)^T v(t) = \bar{v}(t)^T \bar{v}(t) \\ &\leq C e^{-2\sigma_{\min} t}, \end{aligned} \quad \text{where } \sigma_{\min} = \min_{1 \leq i \leq n} |\text{Re}(\lambda_i)|$$

The Luenberger Observer

$$\text{System:} \quad \dot{x} = Ax$$

$$y = Cx$$

$$\text{Observer:} \quad \dot{\hat{x}} = (A - LC)\hat{x} + Ly$$

$$\text{Error:} \quad \dot{e} = (A - LC)e$$

Recall our assumption that $A - LC$ is Hurwitz (all eigenvalues are in LHP). This implies that

$$\|x(t) - \hat{x}(t)\|^2 = \|e(t)\|^2 = \sum_{i=1}^n |e_i(t)|^2 \xrightarrow{t \rightarrow \infty} 0$$

at an exponential rate, determined by the eigenvalues of $A - LC$.

For fast convergence, want eigenvalues of $A - LC$ far into LHP!!

The Luenberger Observer

$$\text{System:} \quad \dot{x} = Ax$$

$$y = Cx$$

$$\text{Observer:} \quad \dot{\hat{x}} = (A - LC)\hat{x} + Ly$$

$$\text{Error:} \quad \dot{e} = (A - LC)e$$

Observer transfer function:

$$s\hat{X}(s) = (A - LC)\hat{X}(s) + LY(s)$$

$$(Is - A + LC)\hat{X}(s) = LY(s)$$

$$\hat{X}(s) = (Is - A + LC)^{-1}LY(s).$$

The eigenvalues of $A - LC$ are the **observer poles**. We want these poles to be *stable* and *fast*.

Observability and Estimation Error

Fact: If the system

$$\dot{x} = Ax, \quad y = Cx$$

is observable, then we can arbitrarily assign eigenvalues of $A - LC$ by a suitable choice of the output injection matrix L .

This is similar to the fact that controllability implies arbitrary closed-loop pole placement by state feedback.

In fact, these two facts are closely related because CCF is dual to OCF.

Observer Pole Placement in OCF

Consider a single-output system in OCF:

$$\dot{x} = Ax$$

$$y = Cx, \quad y \in \mathbb{R}$$

$$\text{where } A = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & -a_n \\ 1 & 0 & \dots & 0 & 0 & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & -a_2 \\ 0 & 0 & \dots & 0 & 1 & -a_1 \end{pmatrix}, \quad C = (0 \quad 0 \quad \dots \quad 0 \quad 1)$$

Note that A^T has the form of a CCF system matrix, thus:

$$\begin{aligned} \det(Is - A) &= \det((Is - A)^T) = \det(Is - A^T) \\ &= s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n \end{aligned}$$

Now Let's Add an Observer

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_n \\ 0 & 1 & \dots & 0 & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -a_2 \\ 0 & 0 & \dots & 1 & -a_1 \end{pmatrix}$$

$$LC = \begin{pmatrix} \ell_1 \\ \ell_2 \\ \vdots \\ \ell_n \end{pmatrix} \begin{pmatrix} 0 & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 & \ell_1 \\ 0 & 0 & \dots & 0 & \ell_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \ell_{n-1} \\ 0 & 0 & \dots & 0 & \ell_n \end{pmatrix}$$

$$A - LC = \begin{pmatrix} 0 & 0 & \dots & 0 & -(a_n + \ell_1) \\ 0 & 1 & \dots & 0 & -(a_{n-1} + \ell_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -(a_2 + \ell_{n-1}) \\ 0 & 0 & \dots & 1 & -(a_1 + \ell_n) \end{pmatrix}$$

— still in OCF!!

Observer Pole Placement in OCF

$$\dot{x} = Ax, \quad y = Cx, \quad \dot{\hat{x}} = (A - LC)\hat{x} + Ly$$

$$A - LC = \begin{pmatrix} 0 & 0 & \dots & 0 & -(a_n + \ell_1) \\ 1 & 0 & \dots & 0 & -(a_{n-1} + \ell_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -(a_2 + \ell_{n-1}) \\ 0 & 0 & \dots & 1 & -(a_1 + \ell_n) \end{pmatrix}$$

Eigenvalues of $A - LC$ are the roots of the characteristic polynomial

$$\begin{aligned} \det(Is - A + LC) \\ = s^n + (a_1 + \ell_n)s^{n-1} + \dots + (a_{n-1} + \ell_2)s + (a_n + \ell_1) \end{aligned}$$

Key observation: In OCF, each observer gain affects only *one* of the coefficients of the characteristic polynomial, which can be assigned arbitrarily by a suitable choice of ℓ_1, \dots, ℓ_n .

Hence the name **Observer Canonical Form** — convenient for observer design.

Observer Pole Placement

General procedure for any *observable* system:

1. Convert to OCF: $T = \underbrace{\mathcal{O}(\bar{A}, \bar{C})^{-1}}_{\text{new}} \underbrace{[\mathcal{O}(A, C)]}_{\text{old}}$
2. Find \bar{L} , such that $\bar{A} - \bar{L}\bar{C}$ has desired eigenvalues.
3. Convert back to original coordinates: $L = T^{-1}\bar{L}$.

The resulting observer is

$$\dot{\hat{x}} = (A - T^{-1}\bar{L}C)\hat{x} + T^{-1}\bar{L}y$$

In fact, this procedure is not necessary because of duality between controllability and observability!!

Controllability–Observability Duality

Claim: The system

$$\dot{x} = Ax, \quad y = Cx$$

is observable if and only if the system

$$\dot{x} = A^T x + C^T u$$

is controllable.

Proof: $\mathcal{C}(A^T, C^T) = [C^T \mid A^T C^T \mid \dots \mid (A^T)^{n-1} C^T]$

$$= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^T = [\mathcal{O}(A, C)]^T$$

Thus, $\mathcal{O}(A, C)$ is nonsingular if and only if $\mathcal{C}(A^T, C^T)$ is.

Observer Pole Placement, O/C Duality Version

Given an **observable** pair (A, C) :

1. For $F = A^T$, $G = C^T$, consider the system $\dot{x} = Fx + Gu$ (this system is controllable).
2. Use our earlier procedure to find K , such that

$$F - GK = A^T - C^T K$$

has desired eigenvalues.

3. Then

$$\text{eig}(A^T - C^T K) = \text{eig}(A^T - C^T K)^T = \text{eig}(A - K^T C),$$

so $L = K^T$ is the desired output injection matrix.

Final answer: use the observer

$$\begin{aligned}\dot{\hat{x}} &= (A - LC)\hat{x} + Ly \\ &= (A - K^T C)\hat{x} + K^T y.\end{aligned}$$

Combining Full-State Feedback with an Observer

- ▶ So far, we have focused on autonomous systems ($u = 0$).
- ▶ What about nonzero inputs?

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

— assume (A, B) is controllable and (A, C) is observable.

- ▶ In the next lecture, we will learn how to use an observer together with estimated state feedback to (approximately) place closed-loop poles.

