Plan of the Lecture

- **Review**: coordinate transformations; conversion of any controllable system to CCF.

- **Today’s topic**: pole placement by (full) state feedback.

**Goal**: learn how to assign arbitrary closed-loop poles of a controllable system \( \dot{x} = Ax + Bu \) by means of state feedback \( u = -Kx \).

**Reading**: FPE, Chapter 7
State-Space Realizations

\[ \dot{x} = Ax + Bu \]
\[ y = Cx \]

\[ G(s) = C(I s - A)^{-1} B \]

Open-loop poles are the eigenvalues of \( A \):

\[ \det(I s - A) = 0 \]

Then we add a controller to move the poles to desired locations:
Goal: Pole Placement by State Feedback

Consider a single-input system in state-space form:

\[
\begin{align*}
\dot{x} & = Ax + Bu \\
y & = Cx \\
\end{align*}
\]

Today, our goal is to establish the following fact:

If the above system is \textit{controllable}, then we can assign arbitrary closed-loop poles by means of a state feedback law

\[
u = -Kx = -(k_1 \ x_1 + k_2 \ x_2 + \ldots + k_n \ x_n),
\]

where \( K \) is a \( 1 \times n \) matrix of feedback gains.
Review: Controllability

Consider a single-input system \((u \in \mathbb{R})\):

\[
\dot{x} = Ax + Bu, \quad y = Cx \quad x \in \mathbb{R}^n
\]

The **Controllability Matrix** is defined as

\[
C(A, B) = \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix}
\]

We say that the above system is **controllable** if its controllability matrix \(C(A, B)\) is **invertible**.

- As we will see today, if the system is controllable, then we may assign arbitrary closed-loop poles by *state feedback* of the form \(u = -Kx\).
- Whether or not the system is controllable depends on its state-space realization.
Controller Canonical Form

A single-input state-space model

\[ \dot{x} = Ax + Bu, \quad y = Cx \]

is said to be in **Controller Canonical Form (CCF)** is the matrices \( A, B \) are of the form

\[
A = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
* & * & * & \ldots & * & * \\
\end{pmatrix}, \quad B = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
1 \\
\end{pmatrix}
\]

A system in CCF is *always controllable*!!

(The proof of this for \( n > 2 \) uses the Jordan canonical form, we will not worry about this.)
We will see that state feedback design is particularly easy when the system is in CCF.

Hence, we need a way of constructing a CCF state-space realization of a given controllable system.

We will do this by suitably changing the coordinate system for the state vector.
Coordinate Transformations and State-Space Models

\[
\dot{x} = Ax + Bu \quad \xrightarrow{T} \quad \dot{x} = \bar{A}\bar{x} + \bar{B}u
\]
\[
y = Cx \quad \bar{y} = \bar{C}\bar{x}
\]

where \( \bar{A} = TAT^{-1} \), \( \bar{B} = TB \), \( \bar{C} = CT^{-1} \)

▶ The transfer function does not change.
▶ The controllability matrix is transformed:

\[
\mathcal{C}(\bar{A}, \bar{B}) = TC(A, B).
\]

▶ The transformed system is controllable if and only if the original one is.
▶ If the original system is controllable, then

\[
T = \mathcal{C}(\bar{A}, \bar{B}) [\mathcal{C}(A, B)]^{-1}.
\]

This gives us a way of systematically passing to CCF.
Example: Converting a Controllable System to CCF

\[ A = \begin{pmatrix} -15 & 8 \\ -15 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (C \text{ is immaterial}) \]

Step 1: check for controllability.

\[ \mathcal{C} = \begin{pmatrix} 1 & -7 \\ 1 & -8 \end{pmatrix} \quad \text{det} \mathcal{C} = -1 \quad \text{controllable} \]

Step 2: Determine desired \( \mathcal{C}(\bar{A}, \bar{B}) \).

\[ \mathcal{C}(\bar{A}, \bar{B}) = [\bar{B} | \bar{A}\bar{B}] = \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix} \]

Step 3: Compute \( T \).

\[ T = \mathcal{C}(\bar{A}, \bar{B}) \cdot [\mathcal{C}(A, B)]^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix} \begin{pmatrix} 8 & -7 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \]
Finally, Pole Placement via State Feedback

Consider a state-space model

\[ \dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R} \]

\[ y = x \]

Let’s introduce a state feedback law

\[ u = -Ky \equiv -Kx \]

\[ = -(k_1 \quad k_2 \quad \ldots \quad k_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = -(k_1 x_1 + \ldots + k_n x_n) \]

Closed-loop system:

\[ \dot{x} = Ax - BKx = (A - BK)x \]

\[ y = x \]
Pole Placement via State Feedback

Let's also add a reference input:

\[ \dot{x} = Ax + Bu \]
\[ y = x \]

Take the Laplace transform:

\[ sX(s) = (A - BK)X(s) + BR(s), \quad Y(s) = X(s) \]
\[ Y(s) = (Is - A + BK)^{-1}BR(s) \]

Closed-loop poles are the eigenvalues of \( A - BK \)!!
Pole Placement via State Feedback

\[ \dot{x} = Ax + Bu \]
\[ y = x \]

assigning closed-loop poles = assigning eigenvalues of \( A - BK \)

Now we will see that this is particularly straightforward if the \((A, B)\) system is in CCF.

\[
A = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
-a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_2 & -a_1
\end{pmatrix}, \quad B = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{pmatrix}
\]
The Beauty of CCF

\[ A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_2 & -a_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \]

Claim.

\[ \det(I_s - A) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n \]

— the last row of the \( A \) matrix in CCF consists of the coefficients of the characteristic polynomial, in reverse order, with “−” signs.
Proof of the Claim

A nice way is via Laplace transforms:

\[ \dot{x} = Ax + Bu \]

\[ A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_2 & -a_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \]

Represent this as a system of ODEs:

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = x_3 \]
\[ \vdots \]
\[ \dot{x}_n = - \sum_{i=1}^{n} a_{n-i+1} x_i + u \]

\[ \left( s^n + a_1 s^{n-1} + \ldots + a_n \right) X_1 = U \]

char. poly.
... And, Back to Pole Placement

\[
A = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
-a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_2 & -a_1
\end{pmatrix}
\]

\[
BK = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{pmatrix}
(k_1 \ k_2 \ \ldots \ k_n) = 
\begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 0 \\
(k_1 \ k_2 \ k_3 \ \ldots \ k_{n-1} \ k_n)
\end{pmatrix}
\]

\[
A - BK = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
-(a_n + k_1) & -(a_{n-1} + k_2) & -(a_{n-2} + k_3) & \ldots & -(a_1 + k_n)
\end{pmatrix}
\]

— still in CCF!!
Pole Placement in CCF

\[
\dot{x} = (A - BK)x + Br, \quad y = Cx
\]

\[
A - BK = \begin{pmatrix}
0 & 1 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 \\
-(a_n + k_1) & -(a_{n-1} + k_2) & \ldots & -(a_1 + k_n)
\end{pmatrix}
\]

Closed-loop poles are the roots of the characteristic polynomial

\[
\det(I\,s - A + BK) = s^n + (a_1 + k_n)s^{n-1} + \ldots + (a_{n-1} + k_2)s + (a_n + k_1)
\]

**Key observation:** When the system is in CCF, each control gain affects only one of the coefficients of the characteristic polynomial, and these coefficients can be assigned arbitrarily by a suitable choice of \(k_1, \ldots, k_n\).

Hence the name Controller Canonical Form — convenient for control design.
Pole Placement by State Feedback

General procedure for any *controllable* system:

1. Convert to CCF using a suitable invertible coordinate transformation $T$ (such a transformation exists by controllability).

2. Solve the pole placement problem in the new coordinates.

3. Convert back to original coordinates.
Example

Given $\dot{x} = Ax + Bu$

\[ A = \begin{pmatrix} -15 & 8 \\ -7 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

Goal: apply state feedback to place closed-loop poles at $-10 \pm j$.

Step 1: convert to CCF — already did this

\[ T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \rightarrow \quad \bar{A} = \begin{pmatrix} 0 & 1 \\ -15 & -8 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]
Example

Step 2: find $u = -\bar{K}\bar{x}$ to place closed-loop poles at $-10 \pm j$.

Desired characteristic polynomial:

$$(s + 10 + j)(s + 10 - j) = (s + 10)^2 + 1 = s^2 + 20s + 101$$

Thus, the closed-loop system matrix should be

$$\bar{A} - \bar{B}\bar{K} = \begin{pmatrix} 0 & 1 \\ -101 & -20 \end{pmatrix}$$

On the other hand, we know

$$\bar{A} - \bar{B}\bar{K} = \begin{pmatrix} 0 & 1 \\ -(15 + \bar{k}_1) & -(8 + \bar{k}_2) \end{pmatrix} \implies \bar{k}_1 = 86, \bar{k}_2 = 12$$

This gives the control law

$$u = -\bar{K}\bar{x} = -\begin{pmatrix} 86 & 12 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$$
Example

Step 3: convert back to the old coordinates.

\[
u = -\bar{K}\ddot{x} = -\bar{K}T x
\]

— therefore,

\[
K = \bar{K}T
\]

\[
= (86 \hspace{1em} 12) \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}
\]

\[
= (86 \hspace{1em} -74)
\]

The desired state feedback law is

\[
u = \begin{pmatrix} -86 & 74 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]