Plan of the Lecture

- **Review**: state-space notions: canonical forms, controllability.

- **Today’s topic**: controllability, stability, and pole-zero cancellations; effect of coordinate transformations; conversion of any controllable system to CCF.

*Goal*: explore the effect of pole-zero cancellations on internal stability; understand the effect of coordinate transformations on the properties of a given state-space model (transfer function; open-loop poles; controllability).

*Reading*: FPE, Chapter 7
State-Space Realizations

\[ \dot{x} = Ax + Bu \]
\[ y = Cx \]

- a given transfer function $G(s)$ can be realized using infinitely many state-space models
- certain properties make some realizations preferable to others
- one such property is controllability
Controllability Matrix

Consider a single-input system \((u \in \mathbb{R})\):

\[
\dot{x} = Ax + Bu, \quad y =Cx \quad x \in \mathbb{R}^n
\]

The Controllability Matrix is defined as

\[
C(A, B) = \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix}
\]

We say that the above system is controllable if its controllability matrix \(C(A, B)\) is invertible.

- As we will see later, if the system is controllable, then we may assign arbitrary closed-loop poles by state feedback of the form \(u = -Kx\).
- Whether or not the system is controllable depends on its state-space realization.
Example: Computing $\mathcal{C}(A, B)$

Let’s get back to our old friend:

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-6 & -5
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
+ \begin{bmatrix}
0 \\
1
\end{bmatrix} u,
\quad
y = \begin{bmatrix}
1 \\
1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
$$

Here, $x \in \mathbb{R}^2 \implies A \in \mathbb{R}^{2 \times 2} \implies \mathcal{C}(A, B) \in \mathbb{R}^{2 \times 2}$

$$
\mathcal{C}(A, B) = [B \mid AB]
\quad
AB = \begin{bmatrix}
0 & 1 \\
-6 & -5
\end{bmatrix}
\begin{bmatrix}
0 \\
1
\end{bmatrix}
= \begin{bmatrix}
1 \\
-5
\end{bmatrix}
$$

$$
\implies \mathcal{C}(A, B) = \begin{bmatrix}
0 & 1 \\
1 & -5
\end{bmatrix}
$$

Is this system controllable?

$$
det \mathcal{C} = -1 \neq 0 \implies \text{system is controllable}
$$
Controller Canonical Form

A single-input state-space model

\[ \dot{x} = Ax + Bu, \quad y = Cx \]

is said to be in Controller Canonical Form (CCF) is the matrices \( A, B \) are of the form

\[
A = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
\ast & \ast & \ast & \ldots & \ast & \ast \\
\end{pmatrix}, \quad B = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
1 \\
\end{pmatrix}
\]

A system in CCF is always controllable!!

(The proof of this for \( n > 2 \) uses the Jordan canonical form, we will not worry about this.)
CCF with Arbitrary Zeros

In our example, we had \( G(s) = \frac{s + 1}{s^2 + 5s + 6} \), with a minimum-phase zero at \( z = -1 \).

Let’s consider a general zero location \( s = z \):

\[
G(s) = \frac{s - z}{s^2 + 5s + 6}
\]

This gives us a CCF realization

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\
x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u,
\]

\[
y = \begin{pmatrix} -z & 1 \end{pmatrix} \begin{pmatrix} x_1 \\
x_2 \end{pmatrix}
\]

Since \( A, B \) are the same, \( C(A, B) \) is the same \( \implies \) the system is still controllable.

A system in CCF is controllable for any locations of the zeros.
OCF with Arbitrary Zeros

Start with the CCF

\[
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad y = \begin{pmatrix} -z \\ 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

Convert to OCF: \((A \mapsto A^T, B \mapsto C^T, C \mapsto B^T)\)

\[
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -z \\ 1 \end{pmatrix} u, \quad y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

We already know that this system realizes the same t.f. as the original system.

But is it controllable?
OCF with Arbitrary Zeros

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
0 & -6 \\
1 & -5
\end{bmatrix} \begin{bmatrix} x_1 \\
x_2 \end{bmatrix} + \begin{bmatrix} -z \\
1 \end{bmatrix} u,
\]
\[
y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\
x_2 \end{bmatrix}
\]

Let's find the controllability matrix:

\[
C(\bar{A}, \bar{B}) = \left[ \begin{array}{c}
\bar{B} \\
\bar{A} \bar{B}
\end{array} \right] \\
\bar{A} = A^T, \quad \bar{B} = C^T
\]

\[
\bar{A} \bar{B} = \begin{bmatrix} 0 & -6 \\
1 & -5 \end{bmatrix} \begin{bmatrix} -z \\
1 \end{bmatrix} = \begin{bmatrix} -6 \\
-z - 5 \end{bmatrix}
\]

\[
\therefore C(\bar{A}, \bar{B}) = \begin{bmatrix} -z & -6 \\
1 & -z - 5 \end{bmatrix}
\]

\[
\det C = z(z + 5) + 6 = z^2 + 5z + 6 = 0 \text{ for } z = -2 \text{ or } z = -3
\]

The OCF realization of the transfer function

\[
G(s) = \frac{s - z}{s^2 + 5s + 6}
\]
is not controllable when \( z = -2 \) or \(-3\), even though the CCF is always controllable.
Beware of Pole-Zero Cancellations!

The OCF realization of the transfer function

\[ G(s) = \frac{s - z}{s^2 + 5s + 6} \]

is not controllable when \( z = -2 \) or \( -3 \), even though the CCF is always controllable.

Let’s examine \( G(s) \) when \( z = -2 \):

\[ G(s) = \left. \frac{s - z}{s^2 + 5s + 6} \right|_{z=-2} = \frac{s + 2}{(s + 2)(s + 3)} = \frac{1}{s + 3} \]

— pole-zero cancellation!

For \( z = -2 \), \( G(s) \) is a first-order transfer function, which can always be realized by this 1st-order controllable model:

\[ \dot{x}_1 = -3x_1 + u, \quad y = x_1 \quad \rightarrow \quad G(s) = \frac{1}{s + 3} \]
Beware of Pole-Zero Cancellations!!

We can look at this from another angle: consider the t.f.

\[ G(s) = \frac{1}{s + 3} \]

We can realize it using a one-dimensional controllable state-space model

\[ \dot{x}_1 = -3x_1 + u, \quad y = x_1 \]

or a noncontrollable two-dimensional state-space model

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} =
\begin{pmatrix}
0 & -6 \\
1 & -5
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} +
\begin{pmatrix}
2 \\
1
\end{pmatrix} u,
\quad y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\
x_2 \end{pmatrix}
\]

— certainly not the best way to realize a simple t.f.!

Thus, even the *state dimension* of a realization of a given t.f. is not unique!!
Beware of Pole-Zero Cancellations!!

Here is a really bad realization of the t.f.

\[ G(s) = \frac{1}{s + 3}. \]

Use a two-dimensional model:

\[ \begin{align*}
\dot{x}_1 &= -3x_1 + u \\
\dot{x}_2 &= 100x_2 \\
y &= x_1
\end{align*} \]

- \( x_2 \) is not affected by the input \( u \) (i.e., it is an uncontrollable mode), and not visible from the output \( y \)
- does not change the transfer function
- ... and yet, horrible to implement: \( x_2(t) \propto e^{100t} \)

The transfer function can mask undesirable internal state behavior!!
Pole-Zero Cancellations and Stability

- In case of a pole-zero cancellation, the t.f. contains much less information than the state-space model because some dynamics are “hidden.”
- These dynamics can be either good (stable) or bad (unstable), but we cannot tell from the t.f.
- Our original definition of stability (no RHP poles) is flawed because there can be RHP eigenvalues of the system matrix $A$ that are canceled by zeros, yet they still have dynamics associated with them.

Definition of Internal Stability (State-Space Version): a state-space model with matrices $(A, B, C, D)$ is internally stable if all eigenvalues of the $A$ matrix are in LHP.

This is equivalent to having no RHP open-loop poles and no pole-zero cancellations in RHP.
Coordinate Transformations

Now that we have seen that a given transfer function can have many different state-space realizations, we would like a systematic procedure of generating such realizations, preferably with favorable properties (like controllability).

One such procedure is by means of *coordinate transformations*. 
Coordinate Transformations

\[ x \mapsto \bar{x} = Tx, \quad T \in \mathbb{R}^{n\times n} \text{ nonsingular} \]

\[ x = T^{-1} \bar{x} \quad \text{(go back and forth between the coordinate systems)} \]
Coordinate Transformations

For example,

\[
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}
\mapsto
\begin{pmatrix}
    \bar{x}_1 \\
    \bar{x}_2
\end{pmatrix}
= \begin{pmatrix}
    x_1 + x_2 \\
    x_1 - x_2
\end{pmatrix}
\]

This can be represented as

\[
\bar{x} = T x,
\]

where \( T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \)

The transformation is invertible: \( \det T = -2 \), and

\[
T^{-1} = \frac{1}{\det T} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}
\]

Or we can see this directly:

\[
\bar{x}_1 + \bar{x}_2 = 2x_1; \quad \bar{x}_1 - \bar{x}_2 = 2x_2
\]
Coordinate Transformations and State-Space Models

Consider a state-space model

\[ \dot{x} = Ax + Bu \]
\[ y = Cx \]

and a change of coordinates \( \bar{x} = Tx \) (\( T \) invertible).

What does the system look like in the new coordinates?

\[ \dot{\bar{x}} = T \dot{x} = T \dot{x} \] (linearity of derivative)
\[ = T(Ax + Bu) \]
\[ = T(AT^{-1} \bar{x} + Bu) \] (\( x = T^{-1} \bar{x} \))
\[ = \underbrace{TAT^{-1}}_{\bar{A}} \bar{x} + \underbrace{TB}_{\bar{B}} u \]
\[ y = Cx \]
\[ = CT^{-1} \bar{x} \]
Coordinate Transformations and State-Space Models

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]

\[
\begin{align*}
\dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\
y &= \bar{C}\bar{x}
\end{align*}
\]

where

\[
\bar{A} = TAT^{-1}, \quad \bar{B} = TB, \quad \bar{C} = CT^{-1}
\]

What happens to

- the transfer function?
- the controllability matrix?
Coordinate Transformations and State-Space Models

\[
\dot{x} = Ax + Bu \quad \rightarrow \quad \dot{x} = \bar{A}\bar{x} + \bar{B}u
\]

\[
y = Cx \quad \Rightarrow \quad y = \bar{C}\bar{x}
\]

where \( \bar{A} = TAT^{-1}, \quad \bar{B} = TB, \quad \bar{C} = CT^{-1} \)

Claim: The transfer function doesn’t change.

Proof:

\[
\bar{G}(s) = \bar{C}(Is - \bar{A})^{-1}\bar{B}
\]

\[
\quad = (CT^{-1})(Is - TAT^{-1})^{-1}(TB)
\]

\[
\quad = CT^{-1}(TIT^{-1}s - TAT^{-1})^{-1}TB
\]

\[
\quad = CT^{-1}[T(Is - A)T^{-1}]^{-1}TB
\]

\[
\quad = CT^{-1}T(Is - A)^{-1}T^{-1}TB
\]

\[
\quad = \underbrace{C}_{I}(Is - A)^{-1}B \equiv G(s)
\]
Coordinate Transformations and State-Space Models

\[
\dot{x} = Ax + Bu \\
\quad \rightarrow \quad T \\
y = Cx
\]

where \( \bar{A} = TAT^{-1} \), \( \bar{B} = TB \), \( \bar{C} = CT^{-1} \)

The transfer function doesn’t change.

In fact:

- open-loop poles don’t change
- characteristic polynomial doesn’t change:

\[
\det(Is - \bar{A}) = \det(Is - TAT^{-1}) \\
= \det [T(Is - A)T^{-1}] \\
= \det T \cdot \det(Is - A) \cdot \det T^{-1} \\
= \det(Is - A)
\]
Coordinate Transformations and State-Space Models

\[ \dot{x} = Ax + Bu \quad \rightarrow \quad \dot{x} = \tilde{A} \tilde{x} + \tilde{B}u \]
\[ y = Cx \quad \rightarrow \quad y = \tilde{C} \tilde{x} \]

where \( \tilde{A} = TAT^{-1} \), \( \tilde{B} = TB \), \( \tilde{C} = CT^{-1} \)

Claim: Controllability doesn’t change.

Proof: For any \( k = 0, 1, \ldots \),

\[ \tilde{A}^k \tilde{B} = (TAT^{-1})^kTB = TA^kT^{-1}TB = TA^kB \quad \text{(by induction)} \]

Therefore, \( \mathcal{C}(\tilde{A}, \tilde{B}) = [TB \mid TAB \mid \ldots \mid TA^{n-1}B] \)
\[ = T[B \mid AB \mid \ldots \mid A^{n-1}B] \]
\[ = TC(A, B) \]

Since \( \det T \neq 0 \), \( \det \mathcal{C}(\tilde{A}, \tilde{B}) \neq 0 \) if and only if \( \det \mathcal{C}(A, B) \neq 0 \).

Thus, the new system is controllable if and only if the old one is.
Coordinate Transformations and State-Space Models

\[
\dot{x} = Ax + Bu \quad \rightarrow \quad \dot{x} = \bar{A}\bar{x} + \bar{B}u
\]
\[
y = Cx
\]
where \( \bar{A} = TAT^{-1} \), \( \bar{B} = TB \), \( \bar{C} = CT^{-1} \)

Note: The controllability matrix does change:

\[
\begin{align*}
\mathcal{C}(\bar{A}, \bar{B}) &= \underbrace{T}_{\text{coord. trans.}} \, \mathcal{C}(A, B) \\
\uparrow &
\end{align*}
\]

\[
T = \mathcal{C}(\bar{A}, \bar{B}) [\mathcal{C}(A, B)]^{-1}
\]

This is a recipe for going from one controllable realization of a given t.f. to another.

CCF is the most convenient controllable realization of a given t.f., so we want to convert a given controllable system to CCF (useful for control design).
Example: Converting a Controllable System to CCF

Note!! The way I do this is different from the textbook.

Consider \( A = \begin{pmatrix} -15 & 8 \\ -15 & 7 \end{pmatrix}, \ B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) (\( C \) is immaterial).

Convert to CCF if possible.

Step 1: check for controllability.

\[
AB = \begin{pmatrix} -15 & 8 \\ -15 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ -8 \end{pmatrix} \implies C = \begin{pmatrix} 1 & -7 \\ 1 & -8 \end{pmatrix}
\]

\[
\det C = -1 \quad \text{-- controllable}
\]
Example: Converting a Controllable System to CCF

Step 2: Determine desired \( C(\bar{A}, \bar{B}) \).

We need to figure out \( \bar{A} \) and \( \bar{B} \).

For CCF, we must have

\[
\bar{A} = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

so we need to find the coefficients \( a_1, a_2 \).

Recall: the characteristic polynomial does not change:

\[
\det (Is - A) = \det (Is - \bar{A})
\]

\[
\det \begin{pmatrix} s + 15 & -8 \\ 15 & s - 7 \end{pmatrix} = \det \begin{pmatrix} s & -1 \\ a_2 & s + a_1 \end{pmatrix}
\]

\[
(s + 15)(s - 7) + 120 = s(s + a_1) + a_2
\]

\[
s^2 + 8s + 15 = s^2 + a_1s + a_2
\]
Example: Converting a Controllable System to CCF

Step 2: Determine desired $C(\bar{A}, \bar{B})$.

We need to figure out $\bar{A}$ and $\bar{B}$.

For CCF, we must have

$$\bar{A} = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We have just computed

$$\bar{A} = \begin{pmatrix} 0 & 1 \\ -15 & -8 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, the new controllability matrix should be

$$C(\bar{A}, \bar{B}) = [\bar{B} | \bar{A}\bar{B}] = \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix}.$$
Example: Converting a Controllable System to CCF

Step 3: Compute $T$.

Recall: $T = C(\bar{A}, \bar{B}) \cdot [C(A, B)]^{-1}$

\[
C(A, B) = \begin{pmatrix} 1 & -7 \\ 1 & -8 \end{pmatrix}
\]

\[
[C(A, B)]^{-1} = \begin{pmatrix} 1 & -7 \\ 1 & -8 \end{pmatrix}^{-1}
\]

\[
= \frac{1}{-1} \begin{pmatrix} -8 & 7 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & -7 \\ 1 & -1 \end{pmatrix}
\]

\[
C(\bar{A}, \bar{B}) = \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix}
\]

\[
T = \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix} \begin{pmatrix} 8 & -7 \\ 1 & -1 \end{pmatrix}
\]

\[
= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}
\]
In the next lecture, we will see why CCF is so useful.