Problem 1. Without a computer, determine whether or not the following polynomials have any RHP roots:

i)  \( s^4 + 10s^3 + 15s^2 + 20s + 1 \)

ii)  \( s^6 + 2s^5 - 3s^4 + s^3 + s^2 + 3s + 5 \)

iii)  \( s^4 + 10s^3 + 12s^2 + 20s + 1 \)

Solution 1.

i)

\[ \begin{array}{ccccc}
 s^4 & 1 & 15 & 1 \\
 s^3 & 10 & 20 \\
 s^2 & 13 & 1 \\
 s^1 & \frac{250}{13} \\
 s^0 & 1 \\
\end{array} \]

Since there is no change of sign in the second column, then No RHP roots.

ii) There is negative coefficients, then RHP roots exist.

iii)

\[ \begin{array}{ccccc}
 s^4 & 1 & 12 & 1 \\
 s^3 & 10 & 20 \\
 s^2 & 10 & 1 \\
 s^1 & 19 \\
 s^0 & 1 \\
\end{array} \]

Since there is no change of sign in the second column, then No RHP roots.
Problem 2. Consider the unity feedback system in Figure 1. Let the plant’s transfer function be given by:

\[ P(s) = \frac{1}{s^3 + 2s^2 + 2s + 1} \]

Suppose our controller is just constant, i.e. \( K(s) = K \).

Use the Routh-Hurwitz criterion to determine which values of \( K \) stabilize the closed-loop system.

Solution 2.

The closed loop transfer function is

\[ H = \frac{Y}{R} = \frac{KP}{1 + KP} = \frac{K}{s^3 + 2s^2 + 2s + K + 1} \]

In order to satisfy the necessary condition, all coefficients should be positive, i.e. \( K + 1 > 0 \), which is \( K > -1 \). Apply Routh-Hurwitz to check other conditions:

\[
\begin{array}{ccc}
  s^3 & 1 & 2 \\
  s^2 & 2 & K+1 \\
  s^1 & -\frac{1}{2}(K-3) & \\
  s^0 & K + 1 & \\
\end{array}
\]

We want all the coefficients are positive, so \(-\frac{1}{2}(K-3) > 0\) and \( K + 1 > 0 \). So if \( K \) satisfies \(-1 < K < 3\), then the closed-loop system is stable.
Problem 3. Recall that a closed-loop system is of type \( n \) with respect to the reference input if the forward-loop transfer function \( K(s)P(s) \) has a pole of order \( n \) at the origin, or, equivalently, if:

\[
\lim_{s \to 0} [s^n K(s)P(s)] = \text{const} \neq 0
\]

The above equation states that the the limit exists, the limit is finite, and the limit is non-zero.

Assuming the closed-loop system is stable, that means that \( n \) is the lowest degree of a polynomial that cannot be tracked in feedback with zero steady-state error. This is what is commonly referred to as system type, and can be thought of as system type with respect to the reference input. The convention is that ‘system type’ typically refers to system type in this sense.

However, we can think of other formulations of system type. In this problem, we’ll consider a generalization that considers system type with respect to disturbances. Consider the unity feedback configuration with an additive input disturbance \( W \) in Figure 2.

i) Let \( T_{r \to y}(s) \) denote the transfer function from \( R \) to \( Y \).

**Remark:** When our system has multiple inputs and one output, we ignore the other inputs when calculating the transfer function from one input to the output.

Show that the system type with respect to reference inputs is \( n \) whenever:

\[
\lim_{s \to 0} \frac{1 - T_{r \to y}(s)}{s^n} = \text{const} \neq 0
\]

ii) Let \( T_{w \to y}(s) \) denote the transfer function from \( W \) to \( Y \).

We say the system has type \( k \) with respect to disturbance inputs if:

\[
\lim_{s \to 0} \frac{T_{w \to y}(s)}{s^k} = \text{const} \neq 0
\]

Show that if \( T_{w \to y}(s) \) has a zero of order \( k \) at the origin, then the system has type \( k \) with respect to disturbance inputs.

**Remark:** If \( T_{w \to y}(s) \) has a zero of order \( k \) at the origin, then we can write it as \( T_{w \to y}(s) = s^k \frac{A(s)}{B(s)} \) where \( A \) and \( B \) are polynomials with real coefficients such that \( A(0) \neq 0 \) and \( B(0) \neq 0 \).

iii) Show that the system of type \( k \) with respect to disturbance inputs can achieve perfect steady-state disturbance rejection with polynomial disturbances with degree \( m < k \), but not when \( m \geq k \).

iv) Finally, consider the plant:

\[
P(s) = \frac{1}{s^2 + 1}
\]

Determine the system type with respect to disturbances under P-control \((K(s) = K_P)\), PD-control \((K(s) = K_P + K_Ds)\), and PID-control \((K(s) = K_P + K_Ds + \frac{K_I}{s})\).
Solution 3.

i) We want to show:
\[
\lim_{s \to 0} [s^n K(s)P(s)] = \text{const} \neq 0
\]

We know \( T_r \to y = \frac{KP}{1+KP} \). We can break this down into two cases.

First, when \( n > 0 \):

\[
0 \neq c = \lim_{s \to 0} \frac{1 - T_r(s)}{s^n} = \lim_{s \to 0} \frac{1}{s^n + s^n K P}
\]

Since \( 1/x \) is continuous at \( x \neq 0 \), and \( x + y \) is continuous for all \( x \) and \( y \), we can conclude:

\[
\lim_{s \to 0} s^n + s^n K(s)P(s) = \lim_{s \to 0} s^n + \lim_{s \to 0} s^n K(s)P(s) = \lim_{s \to 0} s^n K(s)P(s) = \frac{1}{c} \neq 0
\]

The second case is when \( n = 0 \):

\[
c = \lim_{s \to 0} (1 - T_r \to y(s)) = \lim_{s \to 0} \frac{1}{1 + KP}
\]

By the same continuity reasoning as before:

\[
K(0)P(0) = \frac{1}{c} - 1 < \infty
\]

Thus, when \( n = 0 \)

\[
\lim_{s \to 0} s^n K(s)P(s) = \lim_{s \to 0} K(s)P(s) = \frac{1}{c} - 1 \neq 0 \quad \text{whenever} \quad c \neq 1
\]

We’ve shown arguments for both the case where \( n > 0 \) and \( n = 0 \); hence, the system has type \( n \).

**Remark 1.** As one of your classmates pointed out, when \( c = 1 \), we will actually have \( \lim_{s \to 0} K(s)P(s) = 0 \), so the definition of system type given in the homework doesn’t apply. This arises in the case when the DC gain of the system is actually 0. Typically, we will not care about the step response of systems that don’t respond to steps, so you won’t see this often in practice.

ii) Without loss of generality, we can always assume that \( T_{w \to y}(s) = s^{k'} \frac{A(s)}{B(s)} \) with \( k' \in \mathbb{Z} \) and \( A, B \) polynomials with real coefficient such that \( A(0), B(0) \neq 0 \). Since

\[
\lim_{s \to 0} \frac{T_{w \to y}(s)}{s^k} = \lim_{s \to 0} s^{k'-k} \frac{A(s)}{B(s)} = \frac{A(0)}{B(0)} \lim_{s \to 0} s^{k'-k}
\]

If \( k' > k \),

\[
\lim_{s \to 0} \frac{T_{w \to y}(s)}{s^k} = 0
\]

If \( k' < k \), \( \lim_{s \to 0} \frac{T_{w \to y}(s)}{s^k} \) is not defined. Hence we must have \( k' = k \). In other words, \( T_{w \to y} \) has type \( k \) with respect to disturbances inputs if has a zero of order \( k \) at the origin.

iii) Let \( w(t) = t^m \). Then \( W(s) = \frac{1}{s^{m+1}} \). By Final Value Theorem,

\[
y(\infty) = \lim_{s \to 0} T_{w \to y}(s)W(s)s = \lim_{s \to 0} s^{k-m} \frac{A(s)}{B(s)} = \begin{cases} 0 & \text{if } m < k \\ \frac{A(0)}{B(0)} & \text{if } m = k \\ \text{not defined} & \text{if } m > k \end{cases}
\]

Now, suppose the disturbance is any polynomial \( w(t) = a_0 t^m + a_1 t^{m-1} + \ldots a_{m-1} t + a_m \). By linearity, we can see that this system will achieve perfect steady-state rejection if \( m < k \), but not when \( m \geq k \).
iv) Since \( T_{w \to y} = \frac{P}{1+KP} \).

(a) \( T_P = \frac{P}{1+KP} = \frac{\frac{1}{s+K}}{1+\frac{s}{s+1}} = \frac{1}{s^2+KP+1} \). Since there is no zero at origin, hence type 0.

(b) \( T_{PD} = \frac{P}{1+(KP+KDs)s} = \frac{\frac{1}{s+K}}{1+\frac{s}{s+1}} = \frac{1}{s^2+KDs+(KP+1)} \). Since there is no zero at origin, hence type 0.

(c) \( T_{PID} = \frac{P}{1+(KP+KDs+KI)s} = \frac{\frac{1}{s+K}}{1+\frac{s}{s+1}} = \frac{1}{s^2+KPs+(KP+1)s+KI} \). Since there is a zero at origin, hence type 1.